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CONTENTS.

J. SCHMUTZER: "About the oblique extinction of rhombic crystals". (Communicated by Prof. A. WICHMANN), p. 368.

S. J. DE LANGE: "On ascending degeneration after partial section of the spinal cord". (Communicated by Prof. C. WINKLER), p. 386. (With 2 plates).

O. POSTMA: "Motion of molecule-systems on which no external forces act". (Communicated by Prof. H. A. LORENTZ), p. 390.

W. VOIGT: "On the permissible orders of the axes of symmetry in crystallography". (Communicated by Prof. H. A. LORENTZ), p. 408.

H. KAMERLINGH ONNES and C. BRAAK: "Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between —104° C. and —217° C." (Continued), p. 413. (With one plate).

H. KAMERLINGH ONNES and C. BRAAK: "Isotherms of diatomic gases and their binary mixtures. VII. Isotherms of hydrogen between 0° C. and 100° C.", p. 419.

H. KAMERLINGH ONNES, C. BRAAK and J. CLAY: "On the measurement of very low temperatures. XVII. Determinations for testing purposes with the hydrogen thermometer and the resistance thermometer", p. 422.

H. KAMERLINGH ONNES and C. BRAAK: "On the measurement of very low temperatures. XVIII. The determination of the absolute zero according to the hydrogen thermometer of constant volume and the reduction of the readings on the normal hydrogen thermometer to the absolute scale", p. 429.

J. SCHMUTZER: "On the terms Schalie, Lei and Schist". (Communicated by Prof. C. E. A. WICHMANN), p. 432.

F. M. JAEGER: "On the question as to the miscibility and the form-analogy in aromatic Nitro- and Nitroso-compounds". (Communicated by Prof. A. P. N. FRANCHIMONT), p. 436.

P. ZEEMAN: "Observations of the magnetic resolution of spectral lines by means of the method of FABRY and PEROT", p. 440. (With one plate).

H. KAMERLINGH ONNES: "Isotherms of monatomic gases and their binary mixtures. I. Isotherms of helium between + 100° C. and — 217° C.", p. 445.

$$tg\, bZ = tg\, \nu\, cos\, (x - \gamma).$$

Now

$$cot\, 2y = cot\, (aZ + bZ) = \frac{1 - tg\, aZ\, tg\, bZ}{tg\, aZ + tg\, bZ} = \frac{1 - tg\, \mu\, tg\, \nu\, cos\, (x+\gamma)\, cos\, (x-\gamma)}{tg\, \mu\, cos\, (x+\gamma) + tg\, \nu\, cos\, (x-\gamma)} \quad (1)$$

As is indicated in the figure, here the particular case is considered that the zone-axis lies in the plane forming right angles with the acute bisectrix, so that $\mu + \nu = \pi$. If we let the zone-axis successively make different angles α with OP, varying from 0 to π, and if, at the same time, we let this plane perform a revolution about this axis, then N passes through the whole surface of the globe and consequently the extinction with regard to OZ can be calculated as a function of α and x for each arbitrary section through the crystal.

As $\mu + \nu = \pi$, the formula (1) can be simplified as follows:

$$cot\, 2y = \frac{1 + tg^2\, \mu\, cos\, (x + \gamma)\, cos\, (x - \gamma)}{tg\, \mu\, [cos\, (x + \gamma) - cos\, (x - \gamma)]}$$

from which we derive:

$$cot\, 2y = \frac{-(cos^2\, \mu + sin^2\, \mu\, cos^2\, \gamma) + sin^2\, \mu\, sin^2\, x}{sin\, 2\mu\, sin\, \gamma\, sin\, x} =$$

$$= -\frac{cos^2\, \mu + sin^2\, \mu\, cos^2\, \gamma}{sin\, 2\mu\, sin\, \gamma} \cdot \frac{1}{sin\, x} + \frac{sin^2\, \mu}{sin\, 2\mu\, sin\, \gamma}, sin\, x \quad . \quad . \quad (2)$$

$$= \frac{A}{sin\, x} + B\, sin\, x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now in $\Delta\, ZOA$ $cos\, \mu = sin\, OA\, cos\, \alpha = sin\, V\, cos\, \alpha$

and in $\Delta\, ZPA$ $cos\, \angle AZP = \frac{tg\, PZ}{tg\, AZ}$

or

$$cos\left(\frac{\pi}{2} - \gamma\right) = sin\, \gamma = \frac{tg\, \alpha}{tg\, \mu}.$$

If in (2) we substitute the values in α and V for μ and γ, we get:

$$cot\, 2y = -\frac{1 - sin^2\, \mu\, sin^2\, \gamma}{sin\, 2\mu\, sin\, \gamma} \cdot \frac{1}{sin\, x} + \frac{sin^2\, \mu}{sin\, 2\mu\, sin\, \gamma} \cdot sin\, x =$$

$$= -\frac{1 - tg^2\, \alpha\, cos^2\, \mu}{2\, cos^2\, \mu\, tg\, \alpha} \cdot \frac{1}{sin\, x} + \frac{1 - cos^2\, \mu}{2\, cos^2\, \mu\, tg\, \alpha} \cdot sin\, x =$$

$$= -\frac{1 - sin^2\, V\, sin^2\, \alpha}{sin\, 2\alpha\, .\, sin^2\, V} \cdot \frac{1}{sin\, x} + \frac{1 - sin^2\, V\, cos^2\, \alpha}{sin^2 2\alpha\, sin^2\, V} \cdot sin\, x \quad . \quad . \quad (4)$$

From this form we can deduce what follows. For $x = 0$, y

becomes $= \frac{\pi}{2}$ for all values of α, and the same thing takes place with $\alpha = 0$ for all values of x. Consequently on all planes parallel to the acute or to the obtuse bisectrix the extinction with respect to these bisectrices is straight. If the direction of the mean index of refraction (OR) becomes zone-axis, with $\alpha = \frac{\pi}{2}$, a certain particularity shows itself. For this value of α (4) assumes the following form:

$$\cot 2y = \frac{1}{\underset{(2\alpha=\pi)}{\sin 2\alpha}} \left(- \frac{1 - \sin^2 V}{\sin^2 V} \cdot \frac{1}{\sin x} + \frac{\sin x}{\sin^2 V} \right) =$$

$$= \frac{1}{\underset{(2\alpha=\pi)}{\sin 2\alpha}} \left(- \frac{\cos^2 V}{\sin^2 V . \sin x} + \frac{\sin x}{\sin^2 V} \right) \quad . \quad . \quad . \quad . \quad (5)$$

For $x = 0$ becomes $y = \frac{\pi}{2}$.

For $x = \frac{\pi}{2} - V$ (5) changes into:

$$\cot 2y = \frac{1}{\underset{(2\alpha=\pi)}{\sin 2\alpha}} \left(- \frac{\cos V}{\sin^2 V} + \frac{\cos V}{\sin^2 V} \right) = \frac{0}{0}.$$

y becomes indefinite; the pole N of the plane at this moment coincides with an optical axis.

Finally for $x = \frac{\pi}{2}$ y becomes $= 0°$. So the extinction is $\frac{\pi}{2}$ for a value of x between $0°$ and $\frac{\pi}{2} - V$, next becomes indefinite and remains $0°$ for $x = \frac{\pi}{2} - V$ to $\frac{\pi}{2}$, as the sign for $\cot 2y$ shows.

As to the values of y in general, the following may be observed. In (4), if $\frac{\pi}{2} > V > 0$ is assumed, is always

$$1 \geqq 1 - \sin^2 V \sin^2 \alpha > 0$$

$$1 \geqq 1 - \sin^2 V \cos^2 \alpha > 0$$

For a given value of α $\cot 2y$ keeps the same sign, if x varies between 0 and π; it gets, however, negative values for x between 0 and $-\pi$. If we confine ourselves to a variation of x between the limits 0 and $\frac{\pi}{2}$, then the sign of $\cot 2y$ becomes negative for the

values of α, lying between 0 and $\frac{\pi}{2}$; it becomes positive, however for $\frac{\pi}{2} < \alpha < \pi$, whilst the absolute values of y are equal for two poles, lying symmetrically with regard to the plane RO. The same thing holds good for the extinction on planes, lying symmetrically with regard to the plane OP, so that the isogyres drawn upon the globe will lie symmetrically with respect to the planes RO, OP and also RP. Just as the symmetry with regard to RO and OP is accompanied by a change of sign, so also for the plane RP.

The extinction with regard to the variable zone-axis OZ is easy to reduce to that with respect to the acute bisectrix, as the latter is yielded by $\angle ONc = \frown Qc = \frac{\pi}{2} - y = y'$.

$$y = \frac{\pi}{2} - y'$$

$$cot\ 2y = cot\ (\pi - 2y') = -\ cot\ 2y'.$$

from which follows according to (3)

$$cot\ 2y' = -\frac{A}{sin\ x} - B\ sin\ x \qquad \ldots\ldots\ (6)$$

in which:

$$A = -\frac{cos^2\ \mu + sin^2\ \mu\ cos^2\ \gamma}{sin\ 2\mu\ sin\ \gamma}$$

$$B = \frac{sin^2\ \mu}{sin\ 2\mu\ sin\ \gamma}.$$

For the determination of the greatest extinction with regard to the acute bisectrix with $x =$ constant and a variable angle α, we may set about as follows[1]). If we call $\angle ANO = \psi$, $\angle BNO = \psi'$, we find from the triangles ANO and BNO

$$tg\ \psi = \frac{sin\ \angle AON}{sin\ ON\ cot\ V - cos\ ON\ cos\ \angle AON} =$$

$$= \frac{sin\ V\ cos\ \alpha}{cos\ x\ cos\ V - sin\ x\ sin\ V\ sin\ \alpha}$$

and

$$tg\ \psi' = \frac{sin\ V\ cos\ \alpha}{cos\ V\ cos\ x + sin\ V\ sin\ x\ sin\ \alpha}$$

Now $2y' = \psi - \psi'$

$$tg\ 2y' = tg\ (\psi - \psi') = \frac{-2\ sin^2\ V\ sin\ \alpha\ cos\ \alpha\ sin\ x}{sin^2\ V\ cos^2\ \alpha + cos^2\ V\ cos^2\ x - sin^2\ V\ sin^2\ \alpha\ sin^2\ x},$$

which gives:

[1]) A. Harker, Min. Mag. XIII, 1903, p. 66—67.

$$\cot^2\alpha\,(\sin^2 V+\cos^2 V\cos^2 x)-2\cot\alpha\sin^2 V\sin x\cot 2y'+(\cos^2 x-\sin^2 V)=0$$

$$\cot\alpha=\frac{\sin^2 V\sin x\cot 2y'}{\sin^2 V+\cos^2 V\cos^2 x}\pm$$

$$\pm\sqrt{\left(\frac{\sin^2 V\sin x\cot 2y'}{\sin^2 V+\cos^2 V\cos^2 x}\right)^2-\left(\frac{\cos^2 x-\sin^2 V}{\sin^2 V+\cos^2 V\cos^2 x}\right)}$$

As long as the second term remains smaller than the first, the condition for which being $\cos x>\sin V$, or $x<\frac{\pi}{2}-V$, this equation will yield two positive roots, and accordingly two values of α between 0 and $\frac{\pi}{2}$ will satisfy it at a given value of $2y$ smaller than the maximum. The extinction will have reached the maximum, when the two roots are equal, so if

$$(\sin^2 V\sin x\cot 2y')^2=(\sin^2 V+\cos^2 V\cos^2 x)(\cos^2 x-\sin^2 V)$$

or:

$$\sin 2y'_{max}=\frac{\sin V\,tg\,V}{\cos x\cot x}\quad\ldots\ldots\ldots\quad(7)$$

whilst the corresponding value of α [1]) is found from

$$\cot\alpha_{max}=\frac{\sin^2 V\sin x\cot 2y'_{max}}{\sin^2 V+\cos^2 V\cos^2 x}\quad\ldots\ldots\quad(8)$$

The plane ON, in which lies the corresponding pole, then makes with the plane OP an angle $\left(\frac{\pi}{2}-\alpha\right)$.

If we take for olivine the value $2V=87°$ [2]), this gives according to the above formulas the following figures:

TABLE I.

α	μ	γ	A	B
15°	48°19′ 35″	13°48′ 20″	—4.0852	2.3539
30	53 24 25	25 23 10	—2.1480	1.5708
45	60 52 23	33 51 47	—1.6103	1.6103
60	69 52 7	39 25 0	—1.5708	2.1482
75	79 44 14	42 30 39	—2.3548	4.0868

[1]) $\alpha_{(y'=max)}$ is denoted by α_{max}, wherever it could not give rise to ambiguity

[2]) Min. d. Roches, p. 248.

from which the following extinctions with respect to the acute bisectrix are calculated:

TABLE II.

x	Values of y' at $\alpha =$				
	$\alpha = 15°$	$\alpha = 30°$	$\alpha = 45°$	$\alpha = 60°$	$\alpha = 75°$
15°	1°53′ 6″	3°36′ 38″	4°53′ 14″	5° 8′ 26″	3°32′ 41″
30	4 4 28	7 56 59	11 14 42	12 54 21	10 16 52
45	6 49 59	13 42 48	20 38 46	27 27 29	33 6 56
60	10 14 10	20 52 53	32 32 7	46 20 3	64 40 45
75	13 32 29	27 22 46	41 47 36	57 4 59	73 14 21
90	15 (+18″)	30 (+11″)	45	60 (−4″)	75 (−1″)

The values for y' found by calculation with $x = 90°$, which accordingly represent the limit of extinction with regard to the acute bisectrix on the plane making right angles with the latter, give a measure for the exactitude of the values found. The errors successively amount to $+18''$, $+11''$, $0''$, $-1''$, $-1''$.

The greatest extinction for different values of x with the corresponding angle $\left(\frac{\pi}{2} - \alpha\right)$ are now to be calculated from the formulas (7) and (8); we come to the following result:

x	y'_{max}	$\frac{\pi}{2} - \alpha$
15°	5°13′ 12″	34°36′ 4″
30	12 54 29	29 25 6
45	33 44 35	10 40 21
$\frac{\pi}{2} - V =$ 46°30′	45°	0°

To calculate from

$$\cot a_{max} = \frac{\sin^2 V \sin x \cot 2y'_{max}}{\sin^2 V + \cos^2 V \cos^2 x}$$

the value of a_{max} when $\sin x = 0$, we eliminate y'.

As

$$\sin 2y'_{max} = \frac{\sin V \operatorname{tg} V}{\cos x \cot x}$$

we get

$$\cot \alpha_{max} = \frac{\dfrac{\sin^2 V \sin x \sqrt{1 - \left(\dfrac{\sin V \operatorname{tg} V}{\cos x \cot x}\right)^2}}{\dfrac{\sin V \operatorname{tg} V}{\cos x \cot x}}}{\sin^2 V + \cos^2 V \cos^2 x} =$$

$$= \frac{\sin^2 V \sin x \sqrt{\cos^2 x \cot^2 x - \sin^2 V \operatorname{tg}^2 V}}{\sin V \operatorname{tg} V (\sin^2 V + \cos^2 V \cos^2 x)} =$$

$$= \frac{\sin x \cos V \sqrt{\dfrac{\cos^4 x}{\sin^2 x} - \dfrac{\sin^4 V}{\cos^2 V}}}{\sin^2 V + \cos^2 V \cos^2 x} =$$

$$= \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x} \quad \ldots\ldots\ldots \quad (9)$$

When $x = 0$, is

$$\cot \alpha_{max} = \pm \cos V.$$

From which for olivine follows the value:

$$\left(\frac{\pi}{2} - \alpha\right) = bg\ tg\ (\pm) \cos V =$$

$$= bg\ tg\ (\pm) \cos 43°30'$$

$$= (\pm)\ 35°57'22''.$$

In the following figure these results are graphically represented. The black lines connect the poles of planes with equal positive, the lines in black and white those of planes with equal negative extinction. Herein the angles have been considered positive from the acute bisectrix in the direction of the hands of the clock; negative in the opposite direction.

The curves MM' and NN', going through the optic axes, connect the poles of the planes with the greatest (positive and negative) extinction and with the same inclination with regard to the acute bisectrix. The point in which the curves mentioned intersect an isogyre, has on that isogyre the greatest angulardistance from O. For the rest very little need be added to what is to be read from the figure. It shows clearly that an extinction with regard to the acute bisectrix, which déviates little from 0°, is confined to the immediate neighbourhood of the principal planes of symmetry.

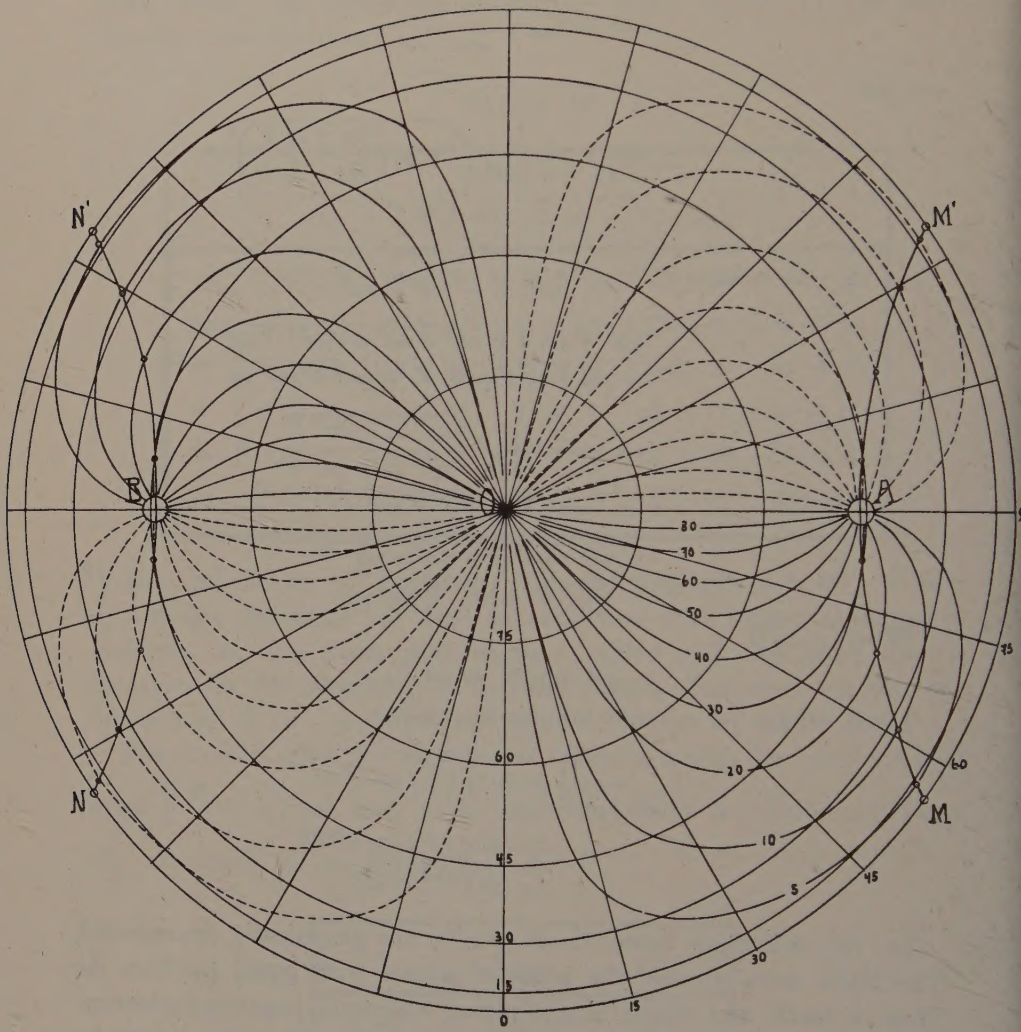


Fig. 2.

For talc ($2V = 13°$) [1]) the following values are calculated in the same way:

[1]) Hintze, Handbuch der Mineralogie, II, 1897, p. 815, cf. Bauer, Pogg. Ann. 1869, CXXXVIII, 368.

	μ	γ	A	B
15°	83°43′ 21″	1°42′ (±1′)	— 156.05	154.32
30	84 22 26	3 15	— 89.846	89.268
35	84 40 45	3 44	— 82.815	82.45
40	85 1 25	4 11	— 78.895	78.718
45	85 24 31	4 36	— 77.628	77.628
50	85 49 38	4 59	— 78.717	78.891
55	86 16 38	5 20	— 82.31	82.672
60	86 45 19	5 38	— 89 268	89.846
65	87 15 5	5 53 30″	— 100.62	101.46
70	87 46 48	6 6	— 120.16	121.38
75	88 19 16	6 16 30	— 154.32	156.05
80	88 52 25	6 24	— 225.49	228.14
85	89 26 4	6 28 30	— 443.61	449.28

to which correspond the extinctions:

x	$\alpha = 15$	$\alpha = 30$	$\alpha = 35$	$\alpha = 40$	$\alpha = 45$	$\alpha = 50$	$\alpha = 55$
15	0° 3′ 4″	0° 4′ 54″	—	—	0° 6′ 9″	—	—
30	0 7 9	0 12 44	—	—	0 14 46	—	—
45	0 8 12	0 26 53	—	—	0 31 19	—	—
60	0 36 55	1 4 59	—	—	1 16 39	—	—
70	1 21 38	2 26 12	2°40′ 54″	2°51′ 34″	2 57 13	2°57′ 48″	2°52′ 14″
75	2 17 13	4 11 16	4 39 53	5 1 35	5 15 26	5 21 1	5 17 6
80	4 23 11	8 24 35	9 31 30	10 33 35	11 23 43	12 1 43	12 24 4
85	9 27 9	19 10 28	22 33 26	26 2 27	29 36 9	33 23 25	37 31 53
90	15°(+53″)	30°(—50″)	—	—	45°	—	—

x	$\alpha = 60$	$\alpha = 65$	$\alpha = 70$	$\alpha = 75$	$\alpha = 80$	$\alpha = 85$	$\alpha = 90$
15	0° 5′ 21″	—	—	0° 3′ 14″	—	—	0°
30	0 12 52	—	—	0 7 28	—	—	0
45	0 27 21	—	—	0 15 56	—	—	0
60	1 7 57	—	—	0 39 55	—	—	0
70	2 41 57	2°26′ 6″	2° 4′ 20″	1 7 40	1° 7′ 27″	0°34′ 26″	0
75	5 1 13	4 35 53	3 58 51	3 9 35	2 12 10	1 7 58	0
80	12 19 51	11 53 14	10 58 49	9 9 38	6 42 42	3 33 47	0
85	41 34 43	46 43 1	53 20 59	59 24 19	67 38 33	78 3 57	90
90	60°(+51″)	—	—	75°(— 8″)	—	—	90°

For the greatest extinction and corresponding angle we find:

x	$y'_{max.}$	$\frac{\pi}{2} - \alpha$
0°	0°	44°48′ 56″
15	0° 6′ 9″	44 48 31
30	0 14 47	44 41 24
45	0 31 21	44 27 2
60	1 16 50	43 38 28
75	5 21 33	39 38 9
$\frac{\pi}{2} - V =$ 83°30′	45°	0°

Figure 3 affords a general view of the results. Suppose the part of the globe-surface, falling outside the parallel-circle of 60° but within the isogyre of 1°, to be equal to the part falling within the same circle outside the isogyre, then it appears that at about $^7/_8$ of the sphere an extinction of less than 1° is observed, so practically a straight extinction. Now the sections, yielding greater extinctions, lie so much in the neighbourhood of the planes, making right angles with the optic axes, that they are for the greater part impracticable for the determination of the direction of extinction. A comparison of figures 2 and 3 shows the result that with rhombic

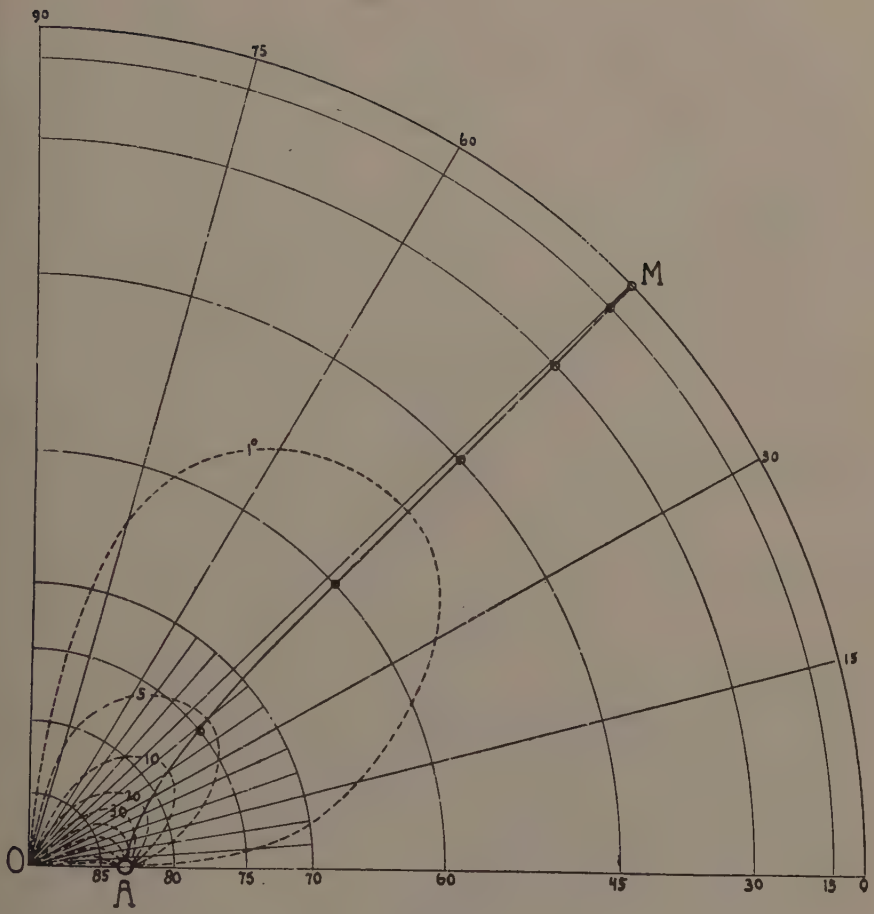


Fig. 3.

crystals with a great axis-angle the oblique, with those with a small axis-angle the straight extinction will predominate, when we have to do with arbitrary sections, as in a rock slice. However in the two cases we as rarely find an absolutely straight extinction. With hexagonal and tetragonal crystals, however, exclusively straight extinction with regard to the optic axis occurs, as for $V=0$ the equation (4)

$$\cot 2y = \frac{1}{\sin^2 V}\left(-\frac{1-\sin^2 V \sin^2 \alpha}{\sin 2\alpha}\cdot\frac{1}{\sin x}+\frac{1-\sin^2 V\,.\cos^2\alpha}{\sin 2\alpha}\sin x\right)$$

always becomes ∞.

In fig. 4 the maximum extinction as a function of x is represented for one globeoctant; MA_1 refers to talc, MA_2 to olivine, MA_3 to a mineral with an axis-angle $2V=160°$ The values OA_1, OA_2

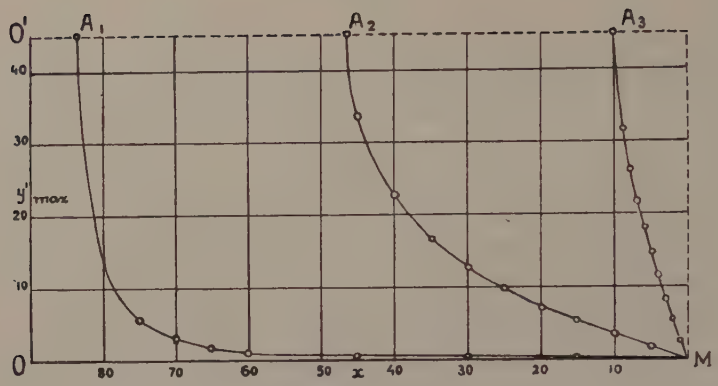


Fig. 4.

and $O'A_3$ give the size of V. The general equation of the curves MA is:

$$\sin 2y'_{max} = \frac{\sin V \, tg V}{\cos x \, \cot x},$$

$$y'_{max} = \tfrac{1}{2} \, bg \sin \left(\frac{\sin V \, tg V}{\cos x \, \cot x} \right)$$

from which

$$\frac{dy'_{max}}{dx} = \frac{\sin V \, tg V}{2 \sqrt{1 - \left(\frac{\sin V \, tg V}{\cos x \, \cot x} \right)^2}} \cdot \left(\frac{1 + 2tg^2 x}{\cos x} \right) =$$

$$= \frac{\sin V \, tg V \, (1 + 2tg^2 x) \, \cot x}{2 \sqrt{\cos^2 x \, \cot^2 x - \sin^2 V \, tg^2 V}} =$$

$$= \frac{\sin^2 V}{2} \cdot \frac{1 + \sin^2 x}{\cos x \sqrt{\cos^4 x \, \cos^2 V - \sin^2 x \, \sin^4 V}} \quad \ldots \ldots \quad (10)$$

So for $x = 0$ the direction of the tangent is given by:

$$\frac{dy'_{max}}{dx} = \frac{+}{(-)} \frac{\sin^2 V}{2 \cos V},$$

For $x = \frac{\pi}{2} - V$ by

$$\frac{dy'_{max}}{dx} = \infty$$

as the term under the root-mark becomes $= 0$. So the tangents in A on the curves form right angles with the direction MO.

It further follows from the formula (10) that the rise of the curve for the same value of x grows smaller as the value of V diminishes, as is also shown by fig. 4. If V becomes $= 0$, as in the hexagonal and tetragonal system, then we also have

$$\frac{dy'_{max}}{dx} = 0\,,$$

so that the curve MA coincides with the abscissa-axis MO. Finally with regard to the form of the curve which represents the angle $\left(\frac{\pi}{2} - \alpha_{max}\right)$ as a function of x, it appears already from a comparison of figures 2 and 3, that this curve MA, the axis-angle becoming smaller, gradually approaches the straight line that divides into two equal parts the angle between OA and the normal to it in O.

Indeed (9) yields

$$\cot \alpha_{max} = \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x}\,,$$

$$= tg\left(\frac{\pi}{2} - \alpha_{max}\right) = \frac{\sqrt{\cos^4 x - (\cos^4 x + \sin^2 x \sin^2 V)\sin^2 V}}{\cos^2 x + \sin^2 x \sin^2 V}\,.$$

If V becomes smaller, $tg\left(\frac{\pi}{2} - \alpha_{max}\right)$ increases, and with $V = 0$ reaches the greatest value $\overset{+}{(-)}\,1$, so that then α_{max} becomes $= 45°$. The curves $M'AM$ and NBN' then pass into two straight lines which intersect in O, thus forming right angles, whilst they have shifted 45° with regard to direction AB.

Of great practical importance is the solution of the problem, how

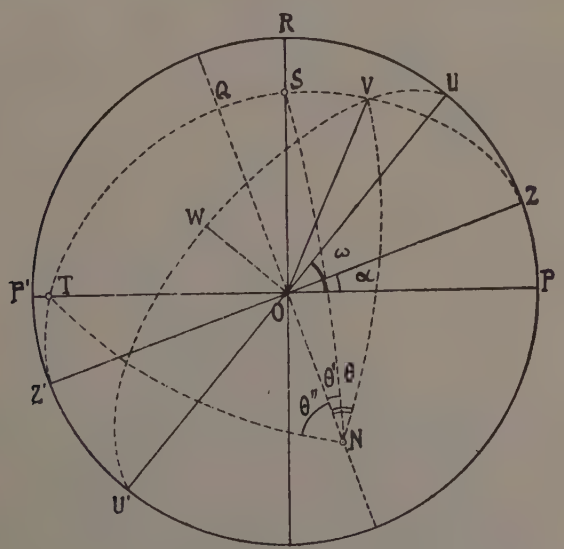


Fig 5.

great the extinction is with regard to the trace of a cleavage plane. For this, if the angle of extinction with regard to the acute bisectrix be known, is only necessary the value of the apparent angle between the trace mentioned and the same bisectrix. One value need only be subtracted from the other.

If ZZ' be the axis of a zone, in which ZQZ_1 represents an arbitrary plane, N the corresponding pole, and be the plane determined again by α and $OQ = x$; if UWU' be an arbitrary cleavage plane, determined by ω and $WO = y$, then VO is the line of intersection of both planes, $VQ = \angle QNV = \theta$, the apparent angle between the acute bisectrix (O) and VO.

Now $VQ = \frac{\pi}{2} - VZ$, and in $\triangle VUZ$ is

$$\angle Z = \frac{\pi}{2} - x$$

$$\angle U = \frac{\pi}{2} + y$$

$$UZ = \omega - \alpha.$$

So

$$\cot VZ = tg\,\theta = \frac{\sin\left(\frac{\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + y\right) + \cos\left(\frac{\pi}{2} - x\right)\cos(\omega - \alpha)}{\sin(\omega - \alpha)}$$

$$= \frac{-\cos x\, tg\, y + \sin x \cos(\omega - \alpha)}{\sin(\omega - \alpha)} \quad \ldots\ldots \quad (11)$$

If we apply this formula to the cleavage planes h^1 (100) and g^1 (010) of olivine, then with

$$h^1(100) \ldots \quad \omega = 0\,,\ y = 0$$

$$g^1(010) \ldots \ \omega = \frac{\pi}{2},\ y = 0$$

and (11) passes into:

$$tg\,\theta' = \sin x\, tg\,\alpha$$

$$tg\,\theta'' = -\sin x \cot\alpha$$

in which θ' and θ'' are successively the apparent angles between the traces of g' (010) and h' (100) on the plane (N).

Now if we think both x and α to vary between 0 and $\frac{\pi}{2}$, we find the following values for θ' and θ'':

α at 0″ (−)	x 0	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$	α at 0′ (+)
0°	indef.	90°	90°	90°	90°	90°	90°	90°
1	0°	86° 8′ 30″	88° 0′ 2″	88°35′ 50″	88°50′ 43″	88°57′ 53″	89	89
2	0	82 18 57	86 0 17	87 10 22	87 41 27	87 55 46	88	88
3	0	78 33 11	84 0 59	85 45 40	86 32 13	86 53 39	87	87
4	0	74 52 5	82 2 19	84 21 9	85 23 2	85 51 34	86	86
5	0	71 19 25	80 4 30	82 56 43	84 13 53	84 49 28	85	85
10	0	55 44 4	70 34 28	75 59 53	78 29 29	79 39 17	80	80
15	0	44 0 26	61 48 38	69 14 48	72 48 29	74 29 45	75	75
30	0	24 8 46	40 33 34	50 46 7	56 18 35	59 7 56	60	60
45	0	14 30 39	26 33 55	35 15 53	40 53 36	44 0 24	45	45
60	0	8 29 56	16 6 8	22 12 28	26 33 54	29 8 50	30	30
75	0	3 58 2	7 37 51	10 43 43	13 3 52	14 30 38	15	15
80	0	2 36 57	5 2 18	7 6 25	8 40 56	9 39 57	10	10
85	0	1 17 49	2 39 17	3 32 24	4 19 58	4 49 49	5	5
86	0	1 2 13	2 0 9	2 49 50	3 27 56	3 51 50	4	4
87	0	0 46 38	1 30 4	2 7 20	2 35 55	2 53 52	3	3
88	0	0 31 4	1 0 1	1 24 52	1 43 56	1 55 55	2	2
89	0	0 15 32	0 30 0	0 42 26	0 51 53	0 57 57	1	1
90	0	0°	0°	0°	0°	0°	0	0

As is also shown in the table

$$\theta' = bg\ tg\ sin\ x\ tg\ \alpha$$

becomes:

for $x = 0$, $\alpha < \frac{\pi}{2}$ equal to 0,

for $x = 0$, $\alpha = \frac{\pi}{2}$ indefinite, and

for $\alpha = 0$ also every time 0, whilst

$$-\theta'' = bg\ tg\ sin\ x\ cot\ \alpha$$

for $x = 0$, $\alpha > 0$ equal to 0,

for $x = 0$, $\alpha = 0$ indefinite, and lastly

for $\alpha = \frac{\pi}{2}$ always becomes 0.

In order to know the extinction of the plane (N) with respect to the trace of g' (010) or h' (100), we combine this table with table II.

For $\frac{\pi}{2} \geqq \alpha \geqq 0$, $\frac{\pi}{2} \geqq x \geqq 0$ the extinctions successively become $(-\theta' + y')$ and $(-\theta'' + y')$; we find the following values:

$$\varphi' = y' - \theta^1.$$

α	$x=0$	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$
0°	0°	0°	0°	0°	0°	0°	0°
15	0	— 2° 4′ 56″	— 3°33′ 23″	— 3°33′ 44″	— 2°49′ 42″	— 0°58′ 9″	0
30	0	— 4 53 18	— 8 9 9	— 8 29 40	— 5 41 1	— 1 46 4	0
45	0	— 9 37 25	—15 19 13	—14 37 7	— 8 21 29	— 2 12 48	0
60	0	—19 0 20	—27 59 13	—23 18 38	— 9 58 32	— 2 2 57	0
75	0	—40 27 45	—51 31 46	—36 7 52	— 8 7 44	— 1 15 24	0
90	indif.	—90°	—90°	—90°	0°	0°	0

When $\alpha = 90°$ and $x = \left(\frac{\pi}{2} - V\right) = 46°30'$ the extinction becomes indefinite; it is here that the transition from 90° to 0° takes place, In the same way we find for

$$\varphi'' = y' - \theta''.$$

α	$x=0$	$x=15$	$x=30$	$x=45$	$x=60$	$x=75$	$x=90$
0°	indefinite	90°	90°	90°	90°	90°	90°
15	0°	45°53′ 32″	65°53′ 6″	76° 4′ 47′	83° 2′ 39″	88° 2′ 14″	90
30	0	27 45 24	48 50 33	64 28 55	77 11 28	86 30 42	90
45	0	19 23 53	37 48 37	55 54 29	73 25 43	85 48 0	90
60	0	13 38 22	29 0 29	49 39 57	72 53 57	86 13 49	90
75	0	7 30 43	17 54 43	43 50 39	77 44 37	87 44 59	90
90	0	0°	0°	0°	90°	90°	90

Also here the extinction for $\alpha = 90°$ and $x = \left(\frac{\pi}{2} - V\right) = 46°30'$ becomes indefinite.

The shape of the φ-isogyres is represented in fig. 6. The black lines refer to φ'', those in black and white to φ'; φ'' gives the value of the positive extinction with regard to the trace of h' (100), φ'

that of the negative extinction with reference to the trace of g' (010) on the plane (N).

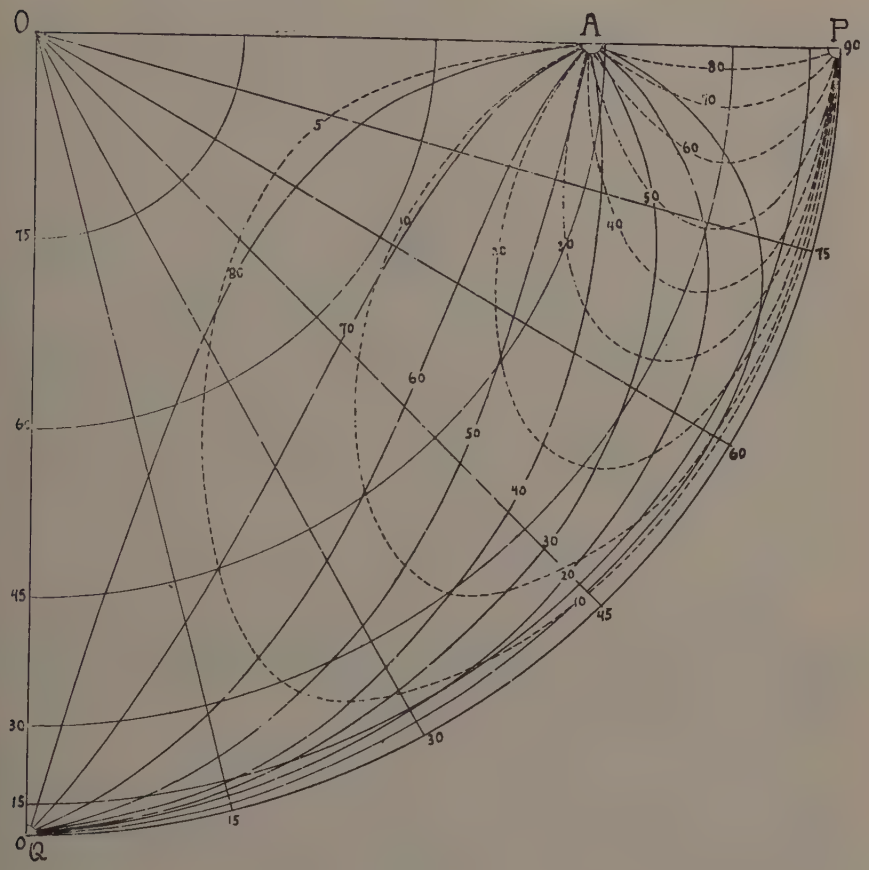


Fig. 6.

The figure is with regard to the axes OP and OQ quite symmetrical again, in the same way as fig. 2 shows for the isogyres with respect to the acute bisectrix, because the points here correspond to the value:

$$\varphi' = \tfrac{1}{2}\, bg\, cot \left(\frac{1 - sin^2 V sin^2 \alpha}{sin\, 2\alpha \,.\, sin^2 V} \,.\, \frac{1}{sin\, x} + \frac{1 - sin^2 V cos^2 \alpha}{sin\, 2\alpha\, sin^2 V} \,.\, sin\, x \right) -$$

$$- bg\, tg\, (sin\, x\, tg\, \alpha) = y' - \theta'.$$

$$\varphi'' = \tfrac{1}{2}\, bg\, cot \left(\frac{1 - sin^2 V sin^2 \alpha}{sin\, 2\alpha \,.\, sin^2 V} \,.\, \frac{1}{sin\, x} + \frac{1 - sin^2 V cos^2 \alpha}{sin\, 2\alpha\, sin^2 V} \,.\, sin\, x \right) +$$

$$+ bg\, tg\, (sin\, x\, cot\, \alpha) = y' - \theta''.$$

Now the signs of y', θ' and θ'' remain unchanged for $0 < x < \pi$ and for α varying between 0 and $\frac{\pi}{2}$, so that also here the sign of the extinction in adjoining globe octants will be alternately positive and negative. The points in which equivalent isogyres in the systems φ' and φ'' intersect, indicate the places of the poles of the sections, in which symmetrical extinction with respect to the cleavage planes h' (100) and g' (010) visible in the slice is observed. So there

$$\varphi' + \varphi'' = 2y' - (\theta' + \theta'') = 0.$$

$$2y' = \theta' + \theta''$$

The curve $OA = V$. If the axis-angle diminishes, A gradually approaches O; the isogyres φ' and φ'' approach a symmetrical direction with regard to the axes OP and OQ, so that the curve, connecting their points of intersection, draws nearer and nearer to the straight line, and at last, when V has become $= 0$, and A coincides with O, passes into the straight line which divides the angle POQ into two equal parts. For $V = 0$ y' becomes $= 0$, so:

$$\theta' + \theta'' = 0$$

$$\sin x \, tg \, \alpha - \sin x \, cot \, \alpha = 0$$

$$tg \, \alpha = cot \, \alpha$$

so that the geometrical place of the points of intersection of the isogyres φ' and φ'', i.e. that of the points of symmetrical extinction, is represented by the line

$$\alpha = 45°.$$

Anatomy. — *"On ascending degeneration after partial section of the spinal cord."* By Dr. S. J. DE LANGE. (Communicated by Prof. C. WINKLER).

(Communicated in the meeting of November 30, 1907).

The following researches have been made with the purpose of investigating whether there exist any connections between the spinal cord and the ascending fasciculus longitudinalis dorsalis, and on the other hand to ascertain once more the course of the ascending anterolateral fascicle of GOWERS and its relation to the dorso-lateral fasciculi of the cerebellum.

From the extensive literature on this subject I will but rarely quote something, whenever the results obtained by others do not accord with my observations. The drawings are taken after four

animals used in the experiments: a full-grown rabbit, in which a hemisection was made at the origin of the medulla oblongata (fig. 1) and three young cats, on which lesions were produced in different portions of the spinal cord (fig. 2, 3 and 4), indicated by C^7 C^4 and D^4.

The central nervous system of these animals was treated by the MARCHI-method, and afterwards cut into serial sections 25 μ thick.

In all these animals a plainly visible degeneration was found after the operation in the tractus spino-cerebellaris anterior or the fascicle of GOWERS. In the dorso-lateral fasciculus there is likewise found a certain degree of degeneration, but far less intense, excepting only in the first animal on which the hemisection was made near the origin of the medulla oblongata. For in this latter case there is found a very compact degeneration of the corpus restiforme, being caused however not only by spino-cerebellar fibres, but likewise and for a great part by bulbo-cerebellar fibres, that were injured when the lesion was produced [1]).

Figures 5—11 represent the degeneration as observed in my preparations. In the first sections we see the dorsal and the ventral spino-cerebellar system still undivided and taking a longitudinal course. Gradually the fibres of the dorso-lateral portion are deviating from the longitudinal direction, whilst the ventral portion still continues to follow it. In a slow spiral-line the dorsal fibres run towards the corpus restiforme; consequently by and by several important portions of the medulla oblongata are found to be encompassed between the two parts of the lateral fasciculi ad cerebellum. This fact becomes most plainly evident in fig. XV, taken from the first cat, on which an almost complete section had been made at C^7. There being no degeneration of the olivo-cerebellar tract in this case, the demarcation between corpus restiforme and antero-lateral fascicle is much more distinct.

I have represented from both these animals the radiation of the corpus restiforme into the cerebellum (dorsal portion of the lateral fasciculi to the cerebellum), and that of the fascicle of GOWERS (antero-lateral portion of the lateral fasciculi to the cerebellum): fig. 10 and 11, 17 and 18. The first radiation is produced through the pedunculus ad cerebellum inferior, that, of the fascicle of GOWERS through the pedunculus ad cerebellum superior.

In order to reach this latter the fibres of the tractus spino-cerebel-

[1]) To understand this rightly, the exact place of lesion must be more accurately circumscribed. The lesion was made just underneath the most caudal root of the hypoglossus, and directed somewhat obliquely upward. Haemorrhage was still to be traced in those sections where the olivary body shows the largest profile.

laris anterior, that continued for a long while their longitudinal course, begin to assume an oblique dorsal direction in the region of the oliva superior and at the first appearance of fibres belonging to the corpus trapezoides. They cross the fibres of the corpus trapezoides in dorso-lateral direction, in consequence of which they present in the sections a peculiar distribution by layers (fig. 9). Until the place of exit of the trigeminus their situation remains nearly unaltered, the fibres lying only more closely together, till they take a sudden turn in dorsal direction, somewhat frontal from the trigeminus, thus joining the course of the lateral ribbon (laqueus lateralis) and staying for the greater part at its outside. Through the pedunculus ad cerebellum the greater part of the fibres now reach the cerebellum, where they take a retrograde direction, extending in the shape of a fan into the vermis inferior. Only a few fibres follow the course of the remaining portion of the lateral laqueus, and reach the corpus quadrigeminum posticum of the same side. Some authors assert that fibres may be traced likewise unto the corpus quadrigeminum anticum (Thomas, Wallenberg, Bianchi, Collier and others), but van Gehuchten, Edinger and Mott agree on this point with Loewenthal, who has been the first to give an accurate description of the fascicle of Gowers.

As I remarked before, the image of the lesions of the spinal cord presents differences only as regards the dorsal portion of the lateral fasciculi to the cerebellum, the degeneration found in this portion being far less intense than it was in the case, where the section was made near the origin of the medulla oblongata. This result, however, must be ascribed not only to the absence of degenerated bulbo-cerebellar fibres, but partly also to the fact that a great number of fibres from the dorsal fascicle find a provisory termination at the passage of the spinal cord into the myelencephalon, as is shown by the preparations. Whilst therefore in the ventral portion we find almost without exception only long fibres. in the dorsal part of the lateral fasciculi of the cerebellum long and short fibres are intermixed, and very probably the connection is composed for the greater part of two neurones.

In all my preparations, some degeneration is observed likewise in the fasciculus longitudinalis posterior. This receptacle of numerous fibres, ascending and descending ones, originating in very different portions of the nerve-trunk, shows degeneration over the whole of its length, and it seems as if from this fascicle all nuclei of motor nerve-fibres are provided with afferent fibres, the Marchi-granulae being found even in the motor roots. In this paper I will not digress longer on the possible significance of this Marchi-granulation, though

perhaps it may be of very great importance. Other methods of investigation will be more efficient in bringing this problem nearer to its solution. In fig. 19 and 20 this granulation is represented, as it appeared in a cat after a central lesion of the 4th dorsal segment (fig. 3). Yet I think I am fully justified in concluding from my preparations to the existence of long ascending fibres, that have their origin probably somewhere near the central canal. In the spinal cord these fibres then continue their course in the anterior fascicle, part of them also in the anterolateral fascicle (fig. 12). The former ones turn more centralward at the origin of the medulla oblongata, and issue in the fasciculus longitudinalis dorsalis. The second ones take a bent near the most caudal root of the hypoglossus, and proceed along the course of that root centralward to the fasciculus longitudinalis (fig. 13, 14 and 25). These ascending fibres are still found even in the nucleus of the N. oculimotorius, as is shown in fig. 26.

Ramon y Cajal also assumes the existence of these fibres, v. Gehuchten however has found ascending degeneration only after a lesion of the nucleus of Deiter, but he does not mention whether he has made observations on sections obtained after central lesions of the spinal cord. In the laboratory of van Gehuchten similar experiments were made by Lubouschine, by the injection of a drop of water. But it was his purpose to destroy part of the posterior horn in order to trace the origin of the anterolateral fascicle. He found this fascicle to be degenerated, but there did not exist any degeneration of the fasciculus longitudinalis, because the central portion of the spinal cord remained uninjured.

After the hemisection near the origin of the medulla oblongata, there appeared also degeneration of the tractus bulbo-cerebellaris, which takes its course from the olivary body towards the periphery of the corpus restiforme as fibrae arcuatae internae and externae. The greater part of these fibres are crossed, still a few of them originate on the same side.

Besides these fibres there are found in the basal portion of the formatio reticularis a great number of degenerated fibres, ascending longitudinally and probably belonging to the secondary sensible tract (fig. 7—11). Part of these fibres transgress the median line and seek the median ribbon (laqueus medialis) of the non-injured side, the rest of them are united near the oliva superior and continue their course in the laqueus lateralis more centralward than the fibres of the fascicle of Gowers. They may be traced into the corpus quadrigeminum posterius.

The significance of the granulation in the corpus trapezoides, which

was to be observed in all cases, did not become clear to me, still I will not pass in silence the fact of the constant appearance of this granulation.

As the summary of my results, I find that after onesided lesion of the spinal cord an ascending degeneration is observed in the following systems:

1. The homo-lateral posterior columns, where it may be traced as far as into the nuclei of GOLL and BURDACH.
2. The lateral fasciculi to the cerebellum,
   a. the dorsal portion almost without exception only on the operated side,
   b. the antero-lateral portion on both sides, but still principally on the operated side.
3. The fasciculus longitudinalis dorsalis on both sides.
4. The corpus trapezoides on both sides (?)

The descending degeneration is represented in figures 21, 22, 23 and 24. It includes:

1. The anterior columns, principally on the operated side, probably centrifugal fibres from the fasc. long. dors.
2. The pyramidal lateral fasciculus on the operated side.
3. The tractus rubro-spinalis, in the lateral columns (VAN GEHUCHTEN).
4. De tractus vestibulo-spinalis, frontal of the anterior horn (EDINGER).
5. Fibres in the posterior columns, being situated partly along the sulcus longitudinalis posterior, and partly along the entering posterior roots, to all probability presenting a homologon to the oval area and the comma of SCHULTZE.

Physics. — "*Motion of molecule-systems on which no external forces act.*" By Dr. O. POSTMA. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the Meeting of November 30, 1907).

§ 1. Up till now two ways have been mainly followed to show that a gas mass left to itself, on which no external forces act, in consequence of the collisions of the molecules will finally pass into a state, in which the molecules are probably about uniformly distributed over the vessel and possess MAXWELL's distribution of velocities.

The first is the method of BOLTZMANN, who assuming that the density all through the vessel is already the same, and further starting from the assumption that there is no regular arrangement of the molecules as regards the velocity, demonstrates that a certain

Fig. I. HEMISECTIO MEDULLA OBL.

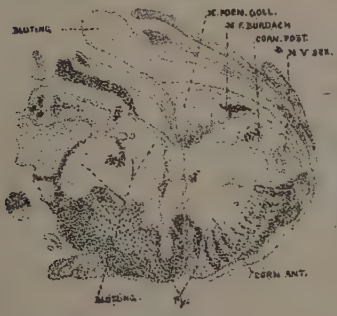


Fig. II. HEMISECTIO IN C. 7

Fig. III. LAESIO CENTR. v. D. 4.

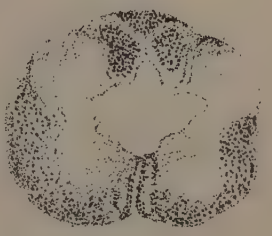


Fig. IV HEMISECTIO IN C. 4.

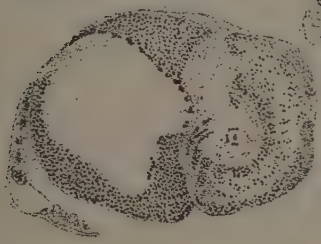


Fig. V

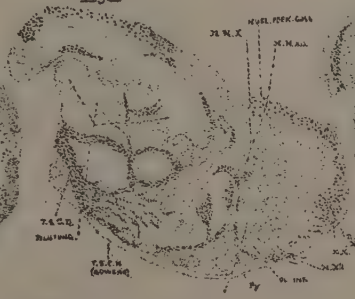


Fig. VI

Fig. VII.

Fig. VIII.

Fig. IX

Fig. X

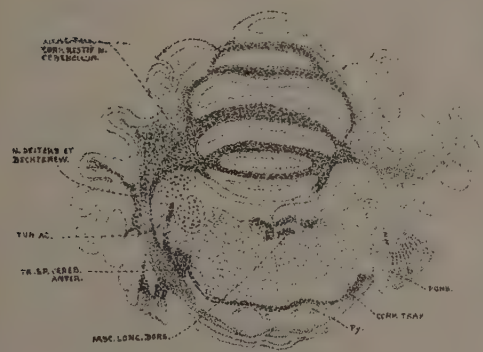


Fig. XI

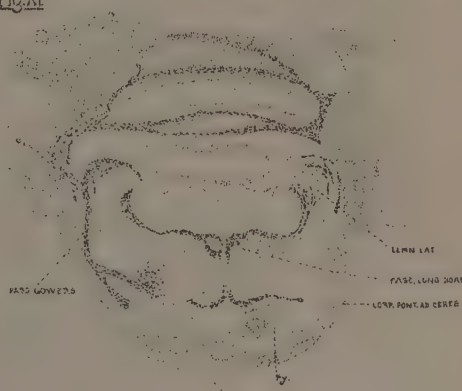


Fig. XII.

Fig. XIII

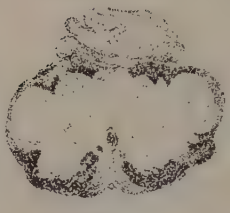


Fig. XIV.

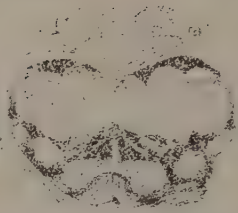


Fig. XV

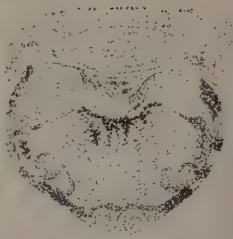


Fig. XVI. CORPUS TRAPEZOIDES.

Fig. XVII.

Fig. XVIII.

Fig. XIX.

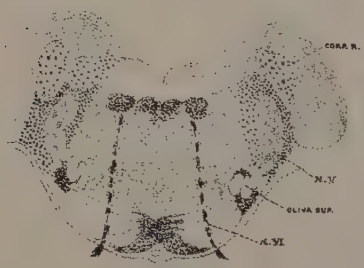


Fig. XX

Fig. XXI

Fig. XXII.

Fig. XXIII

Fig. XXIV

Fig. XXV.

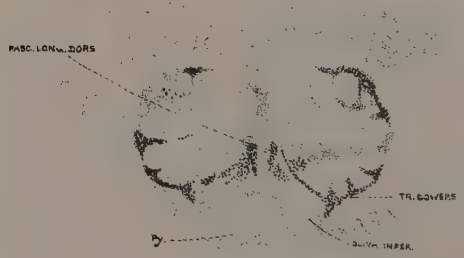
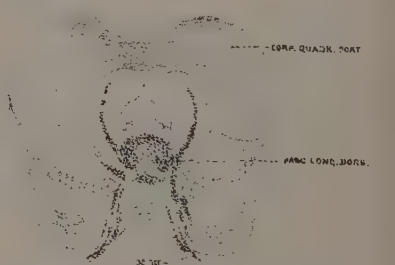


Fig. XXVI.



quantity $H = \int f . lf . d\omega$ constantly decreases by the collisions till the stationary state is reached, with which, as appears, MAXWELL's distribution of velocities exists.

The second is the method half followed by BOLTZMANN, and entirely by JEANS, by which it is demonstrated that on certain hypotheses the state with uniform density and that with Maxwell's distribution of velocities are the most probable. These hypotheses are, as regards the distribution of place, that every time there would be an equal chance to any place in the vessel for every molecule; with regard to the distribution of velocities that there would every time be an equal chance that the point of velocity of a molecule would get into any arbitrarily chosen volume-element, in which we should finally have to reckon with the fact that the total energy has a certain definite value.

I have tried to show [1]) that there is something contradictory in this, which might be avoided by assuming that the gasmass is arbitrarily chosen from a microcanonical ensemble, of which all the systems possess the energy which the gasmass must have. For in this all the combinations of place and all the combinations of velocities with the same energy are equally numerous, and so we have the same chance to hit upon them for the system chosen.

So another proof for the above mentioned result is furnished, when we show that an arbitrary ensemble of systems with the same energy, left to itself, passes into a microcanonical ensemble. GIBBS endeavours to demonstrate this in the XIIth Chapter of his "Statistical Mechanics"; the reasoning is made clearer by LORENTZ [2]), though the latter goes no further than calling the assumption that we should finally get a microcanonical distribution, very plausible.

However in a recent paper [3]) POINCARÉ called attention to a property in the light of which, in my opinion, the above reasoning is no longer tenable. There POINCARÉ shows, namely, that the quantity $S = \int P \log P dx_1 \ldots dx_n$ (in which $x_1 \ldots x_n$ represent the variables which determine every system of a certain ensemble, and $P = \frac{D}{N}$ the coefficient of probability, the integration being extended over the

[1]) These Proc. Febr. 21, 1906 and Jan. 24, 1907.

[2]) "Über den zweiten Hauptsatz der Thermodynamik"; Abhandlungen über Theoretische Physik, Leipzig 1906, p. 289.

[3]) "Réflexions sur la Théorie cinétique des gaz"; Journal de Physique, 1906, p. 369.

whole region occupied by the ensemble) is constant if the external circumstances are unchanged and the relation $\Sigma \frac{\partial X_i}{\partial x_i}$ exists (for which $X_i = \frac{dx_i}{dt}$). In this POINCARÉ takes as variables the coordinates and velocity-components of the molecules; so the quantity referred to above differs from the quantity η introduced by GIBBS only by a constant factor.

GIBBS shows that in a canonical ensemble $-\eta$ has the properties of the entropy, POINCARÉ calls the quantity S itself entropy, also for an arbitrary ensemble. Hence this quantity will have to decrease, where $-\eta$ increases.

The property in question may be derived as follows: P has the properties of a density, so:

$$\frac{\partial P}{\partial t} = -\Sigma \frac{\partial P X_i}{\partial x_i} = -\Sigma X_i \frac{\partial P}{\partial x_i} \left(\text{as } \Sigma \frac{\partial X_i}{\partial x_i} = 0 \right).$$

Now, an arbitrary function of P will also behave as a density. Namely:

$$\frac{\partial f(P)}{\partial t} = f'(P) \frac{\partial P}{\partial t} = -f'(P) \Sigma X_i \frac{\partial P}{\partial x_i} = -f'(P) \Sigma X_i \frac{\partial P}{\partial x_i} - f(P) \Sigma \frac{\partial X_i}{\partial x_i} =$$

$$= -\Sigma X_i f'(P) \frac{\partial P}{\partial x_i} - \Sigma f(P) \frac{\partial X_i}{\partial x_i} = -\Sigma \frac{\partial [X_i f(P)]}{\partial x_i}.$$

So the $f(P)$ also satisfies an equation of the same form as the P itself, which equation represents the extension of the wellknown equation from hydrodynamics:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u}{\partial x} + \frac{\partial \varrho v}{dy} + \frac{\partial \varrho w}{dz} = 0$$

to a space of n dimensions.

Now $S = \int P \log P d\tau$, (in which $d\tau = dx_1 \ldots dx_{n_1}$) $= \int f(P)\, d\tau$, is the integral of such an $f(P)$, integrated over the whole extension occupied by the ensemble. To ascertain the change of S with the motion of the ensemble, we must every time integrate over the variable space (though constant in size), over which the phases extend or where P and also $f(P)$ have values. So we can perfectly compare $\frac{dS}{dt}$ with the increase in time of a quantity of liquid, taken over all the places where it is. This increase, however, is equal to zero.

However, when $S = \int P \log P \, dx_1 \ldots dx_n$ is constant, the ensemble cannot move in the direction to the microcanonical distribution. For then P would become constant all over the phase-extension in course of time, and so the integral would get a minimum value.

Now, however, the question suggests itself: when the function S or η is constant, is there not another quantity characterising the ensemble, which by its variation in a certain direction indicates the motion of the ensemble in the direction of the uniform distribution of place and MAXWELL's distribution of velocities? For this motion is hardly open to doubt, and in a special case such a function has been found for one system by BOLTZMANN in the quantity H.

POINCARÉ supposes he has found such a function in his "entropie grossière", a quantity of the same form as S, but in which the elements of the area over which the summation is made, are not taken infinitely small, but so small that practically we cannot distinguish between systems lying within the same element. This quantity may, therefore, be represented by $\Sigma \Pi \log \Pi \, . \, \sigma$, in which σ represents the element and Π the mean density in it. In contradistinction with this entropy S is called the "entropie fine", and it may easily be shown that the "entropie grossière" is always smaller than the "entropie fine". It is less easy to see that the "entropie grossière" gradually decreases, nor does POINCARÉ prove this. For it is not easy to see that the quantity $S = \iint P \log P \, dl \, d\omega$, of which he tries to prove in some cases that it has decreased, represents an "entropie grossière", while the proof too rests on an assumption which is unjustified in my opinion. It is true that we shall demonstrate further on, that there are quantities of this form which decrease, but for them the name of "entropie grossière" is not very appropriate, as the elements of the extension over which the summation is made are just as well infinitely small, though of lower order of magnitude than the original elements.

§ 2. A very suitable introduction in the theory of gases is supplied by the *problem of the small planets*[1]) repeatedly treated by POINCARÉ. There the problem is discussed what in course of time the distribution along the ecliptic will become of a number of small planets, which at some time were placed in their orbit in such a way that chance has decided at least the distribution of the velocities. POINCARÉ shows

[1]) Cf. l. c. and also: "Calcul des Probabilités", Paris 1896 and "La Science et l'Hypothèse" Paris 1904.

that if the number is large, and the planets do not interfere with each other, in the long run the planets will most likely get about uniformly scattered over the ecliptic.

If we should wish to treat the problem in exactly the same way as GIBBS, we should have to consider an ensemble of systems each consisting of n planets. As, however, the planets do not interfere with each other, we may also take an ensemble of systems of only one planet, in which case the ensemble represents all possibilities which may occur in the placing of a planet. When now such an ensemble, satisfying certain simple conditions, gradually spreads uniformly over the ecliptic, there is for every planet chosen at random from this ensemble, finally an equal chance to any place of the ecliptic, so that, if we have to choose a planet from such an ensemble n times, they will most probably be distributed about uniformly over the ecliptic, if n is large.

It is assumed that the orbits are circular, and lie in the plane of the ecliptic, so that every planet is determined by the variables l (length) and ω (angular velocity), in which ω is constant, and $l = l_0 + \omega t$, if also larger angles than 2π are admitted. The function S (POINCARÉ's entropie fine) is, accordingly, here $\iint P \log P \, dl d\omega$ integrated over all the phases.

As X_1 corresponding to l is equal to ω, and X_2 corresponding to ω is equal to 0, here $\sum \frac{\partial X_i}{\partial x_i} = \frac{\partial \omega}{\partial l} = 0$, so the function S remains constant.

Yet the ensemble approaches uniform distribution over the ecliptic, which, however, is an altogether different thing from the density P becoming constant. Then S would, of course, decrease (it must be observed that $\iint P dl d\omega$ and $\iint dl d\omega$ remain also constant). This approach to uniform distribution is perhaps most readily seen, when we consider only that part of the ensemble that had originally a length between l_0 and $l_0 + dl_0$, the angular velocities lying between ω_1 and ω_2. This part of the ensemble, being originally found in one point of the ecliptic, will get disintegrated by the different velocities, and gradually spread over the ecliptic, till finally the ecliptic is taken up a very large number of times. At a definite point of the ecliptic there are now parts, which were originally spread over a large number of elements of the extension, always at equal distances from each other, and it is easy to see, that if the function representing the original density, and its derivative are finite and continuous, the

density along the ecliptic will finally be the same everywhere [1]). The adjoined fig. 1 represents the motion of the ensemble. Every point of the originally horizontal elementary area moves upwards in a vertical direction with constant velocity, so that the extension always occupies a slanting area with an inclination determined by $tg\alpha = t$.

The horizontal areas at distances 2π from each other indicate the parts of the extension, which are in the same point of the ecliptic. Originally these parts have been in parts of the original area at equal distances from each other. These distances become smaller and smaller, the surface elements becoming more numerous and at the same time smaller.

Instead of the constant quantity $S = \iint P \log P dl\, d\omega$ (l thought to be continuous) we get a variable, when we immediately take together the

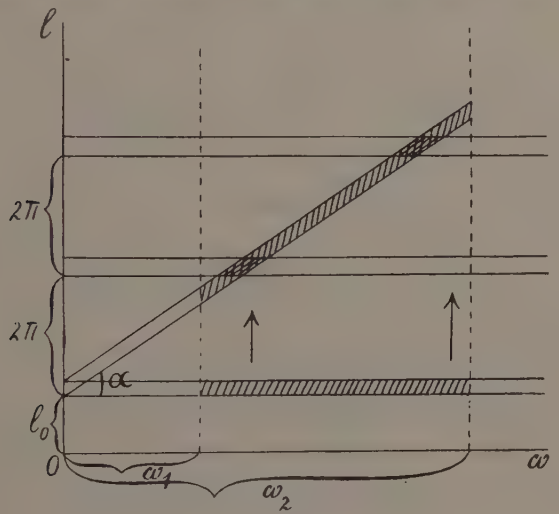


Fig. 1.

surface elements which come to the same thing with respect to the place in the ecliptic. So we get the quantity $S_p = \int_0^{2\pi} P' \log P'\, dl$, in which P' represents the quantity found per unit of length in the conjoint areas, of a width dl and at distances 2π apart, which give the same placing in the ecliptic. So in the case referred to, that originally a horizontal area whose width is dl_0 was occupied, $P'\, dl = \Sigma\, P_i dl_0\, dl \cot \alpha = \Sigma\, P_i \frac{dl_0\, dl}{t}$ so $P' = \frac{dl_0}{t} \Sigma\, P_i$, in which P_i re-

[1]) Cf. Poincaré l.c.

presents the density in each of the points of the original area which now come to the same thing.

Now this P' becomes, as we saw, in course of time a constant and so S_p minimum. This quantity might be called *entropy of place*; "entropie grossière" seems less appropriate, because the elements of extension are infinitely small. This final approach to a minimum must not be taken as a continual decrease. It is easy to see that the original function of density might be chosen in such a way that there are also times, at least in the beginning, at which the densities P' rather diverge than draw nearer to each other. Then the quantity S_p would increase; so $\frac{dS_p}{dt}$ is not like BOLTZMANN's $\frac{dH}{dt}$ negative all through.

If, however, we compare times, in which first an angle 2π, then 4π etc. is occupied, we may say with a pretty high degree of certainty, that S_p has always diminished. If now instead of a horizontal area an arbitrarily chosen ensemble is considered, the above reasoning may be applied for every horizontal elementary area from it. So now too the ensemble is finally uniformly spread over the ecliptic, and the quantity $S_p = \int_0^{2\pi} P' \log P' \, dl$ becomes minimum. Now, however, $P' \, dl = \Sigma \int P \, dl \, d\omega$ or $P' = \Sigma \int P \, d\omega$, every time integrated over all ω's which fall within the horizontal area determined by dl. This P' becomes finally constant. The motion of the ensemble in this more general case is expressed by fig. 2.

The above mentioned inaccuracy in POINCARÉ's reasoning is this: he considers the quantity $S = \iint P \log P \, dl d\omega$, in which he integrates with regard to l from 0 to 2π. So the P from this formula has evidently arisen by summation for the different values of l which come to the same thing, but not by integrating with respect to ω. Hence this P is the sum of the densities of the elements obtained by taking a definite ω, and then successively $l, l + 2\pi$ etc. In fig. 2 these elements are cross-hatched for one value of ω. However, POINCARÉ assumes further that finally for $t = \infty$ this P, or rather this ΣP no longer depends on l, but only on ω [1]). This seems to me incorrect. For then for every vertical elementary area in fig. 2 the sum of the densities in the elements, which are every

[1]) Réflexions sur la Théorie cinétique des gaz; p. 381, p. 385 etc.

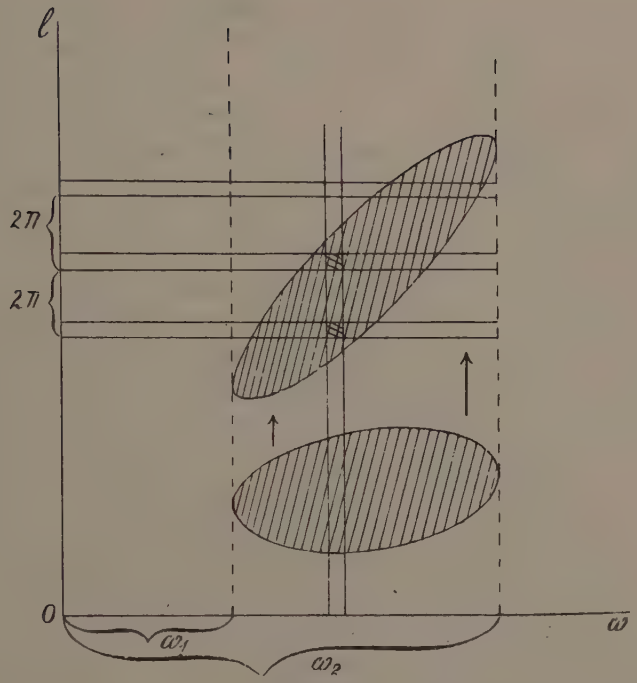


Fig. 2.

time at a distance 2π from each other, would finally have to become constant. This, however is only possible if the number of terms of this sum becomes at last infinitely large, which is by no means the case. Every vertical distance within the extension remains, namely, of the same length, so that the number of elements within each vertical area which are to be taken together, remains always finite. Only when the occurring ω's extend over a finite distance, the number of terms of the sum for $t = \infty$ can also become infinitely large.

A second partial entropy, that *with regard to the velocities* is obtained by taking these elements of the extension together which give the same velocity. So this is here $S_s = \int_{\omega_1}^{\omega_2} P'' \log P'' \, d\omega$, in which $P'' = \int P \, dl$ is integrated along the vertical areas. This entropy, is indeed, also smaller than the "entropie fine", but the difference remains constant, and so also S constant.

§ 3. A transition case from that of the planets to that of the gas molecules is furnished by the case of a gas of one dimension. By this POINCARÉ understands a gas, all the molecules of which move

parallel to each other to and fro between two parallel walls normal to the direction of motion of the molecules. We assume that perfectly elastic collision takes place, and that the dimensions of the molecules may be neglected. Now if we trace the whole way always in one direction, and introduce the distance to one of the walls as variable l, further consider the velocity ω as constantly positive, this case is identical with the preceding one, if we call the distance between the walls π, only with this difference, that now the distances $l, 2\pi - l, 2\pi + l$ etc. come to the same thing as regards the placing of the molecules. So twice as many areas must be taken together as before, and now we have only to integrate from 0 to π. Just as before we have now a quantity $S_\rho = \int_0^\pi P' \log P' dl$, which decreases because $P' = \Sigma \int P \, d\omega$ becomes at last a constant [1]).

Such molecules not interfering with each other in their motion, the case of a continuous ensemble of systems of one molecule each, and that of a real gas of n molecules is pretty much the same, just as for the planets. For a gas of three dimensions this is in general no longer the case in consequence of the collisions.

We meet with another transition case in an ensemble of systems consisting of n planets each. The "entropie fine" is now:

$$S = \int P \log P \, dl_1 \ldots dl_n \, d\omega_1 \ldots d\omega_n,$$

the S_ρ is given by

$$S_\rho = \int_0^{2\pi} \ldots \int_0^{2\pi} P' \log P' \, dl_1 \ldots dl_n,$$

in which $P' = \Sigma \int P \, d\omega_1 \ldots d\omega_n$, integrated with respect to $\omega_1 \ldots \omega_n$ and summed over all the combinations from l_1 to l_n which give the same arrangement of the planets. These combinations are obtained by combining the values l_1, $2\pi + l_1$ etc. with the values l_2, $2\pi + l_2$ etc. to l_n, $2\pi + l_n$ etc. in all ways possible. The number of these combinations, so the number of terms in the summation increases indefinitely during the motion, just as in the preceding cases; so P' becomes constant, and S_ρ approaches the minimum value. If

[1]) Here we might also have taken the coordinate x varying from 0 to π as variable instead of the continuous l. Then the ω shifts every time from + to − and vice versa, and we get the same terms $\int P d\omega$, but now at the same height x.

instead of the planets we imagine molecules moving in a one-dimensional motion, we get: $S_p = \int_{0}^{\pi} \ldots \int_{0}^{\pi} P' \log P' \, dl_1 \ldots dl_n$ in which l_1, $2\pi - l_1$, $2\pi + l_1$ etc. must be taken together in the calculation of P'.

We meet with still another transition case when we consider an ensemble of systems of one molecule each, moving freely in a vessel having the shape of a right-angled parallelopiped with the edges a, b and c. This motion may be thought as composed of three motions parallel to the edges, which we may each treat as in the first transition case. If we call the coordinates with respect to the sides x, y and z, the combinations of x, $2a-x$, $2a+x$ etc. with y, $2b-y$, $2b+y$ etc. and z, $2c-z$, $2c+z$ etc. come to the same thing with regard to place. So we now get the P' by integrating for the P with regard to the components of the velocity, and by then summing for all these combinations. The $S_p = \int_0^a \int_0^b \int_0^c P' \log P' \, dx \, dy \, dz$ decreases again till the minimum value is reached. A molecule chosen at random from the ensemble will have an equal chance to any place in the vessel. When the dimensions of the molecules are not neglected planes take the place of the walls of the vessel at a distance r parallel to them.

If we may disregard the collisions of the molecules inter se of a gas mass, we might always consider each of the n molecules as chosen arbitrarily from such an ensemble, and hence at last these n molecules would probably be distributed over the vessel about uniformly.

§ 4. Finally we shall consider an ensemble of systems consisting of n molecules moving in a vessel having the shape of a right-angled parallelopiped. If we take the coordinates with regard to the side faces as variables, the components of the velocities may also be negative and the representing point of a system may occupy any place within the space

$$\int_r^{a-r} dx_1 \ldots dx_n \int_r^{b-r} dy_1 \ldots dy_n \int_r^{c-r} dz_1 \ldots dz_n \int_{-\infty}^{+\infty} d\xi_1 \ldots d\xi_n \int_{-\infty}^{+\infty} d\eta_1 \ldots d\eta_n \int_{-\infty}^{+\infty} d\zeta_1 \ldots d\zeta_n$$

in which we need only take into consideration that during the motion the kinetic energy or also the Σv^2 remains constant. So we shall now have to examine which parts of the phase extension will give the same arrangement of the molecules, and which the same molecular

velocities, and to ascertain whether the P integrated and summed over these spaces becomes constant in course of time.

It stands to reason that here the problem of the distribution of velocities will be the simplest, because it is modified directly by the impact, and the distribution of place only indirectly.

In agreement with § 2 where we considered an area from the original extension lying between l_0 and $l_0 + dl_0$ in order to examine the placing, we shall take here a part from the extension determined by limits of velocity lying infinitely near each other, but covering a finite part of the $3n$-dimensional space of coordinates. In connection with the condition that $\Sigma v^2 = C$, we take from the $3n$-dimensional space of velocities an element of a spherical shell, whose radius is $\sqrt{\Sigma v^2}$. To this corresponds a prismatic or cylindrical part of the extension, the base of which is represented by the element in question. With regard to the distribution of velocities the points from these and similar prismatic or cylindrical tubes come to the same thing. The elements of the spherical shell represent the projection of the tubes on the space of velocities. Now it remains to investigate whether the quantity of substance, which originally is found above the element mentioned in a given tube, will not finally have spread uniformly over all the tubes, so that the same quantity will be found above every element of the same size. If so, $S_s = \int P' \log P' d\tau$ will again become minimum, if $d\tau$ represents the size of such an element, and $P' d\tau$ the quantity which is projected in $d\tau$.

We may also call the points from the element of the spherical shell the points of velocity of the systems, and the vector, which joins the origin with such a point of velocity, represents all the velocities of the system both with regard to magnitude and to direction; the projections of the vector on the $3n$ axes of the space are the components of the molecular velocities.

The best way of setting forth the gradual uniform dispersion of these points over the mentioned hypersphere is perhaps by availing ourselves of BOREL's mode of representation, and by partially following his method. [1])

BOREL imagines that in the same $3n$-dimensional space in which the coordinates of the molecules are laid out (so that we get in this way a point representing the total arrangement for every system) also the components of velocity are projected starting from the representing point mentioned. The vector then obtained represents the velocity

[1]) "Sur les Principes de la Théorie cinétique des gaz" par EMILE BOREL. Annales de l'Ecole Normale Supérieure IIIe Série, 1906, N°. 1, p. 9.

of the representing point, and we may now examine what takes place with this vector during the motion. It is now obvious that the representing point moves in a space inclosed by surfaces and spaces as $x_1 = r$, $x_1 = a - r$ etc. with which it collides when one of the molecules of the system strikes against a wall of the vessel. When two molecules collide, the representing point strikes against a surface, the equation of which has the form:

$$(x_m - x_n)^2 + (y_m - y_n)^2 + (z_m - z_n)^2 = 4r^2.$$

Now BOREL shows that for this impact the same rules hold as for ordinary collisions, so that the lines along which the point moves before and after impact, lie in the same plane as the normal, the normal dividing the angle of the first two lines into two equal parts. The velocity, too, remains the same According to the above we must now imagine a finite space filled with such representing points, an infinitely small pencil of vectors of velocity starting from each point, mutually equal, and also equal for all the points. Now a number of these representing points strikes against one of the above mentioned surfaces, e.g. with equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 4r^2;$$

this implies then that for these systems the first and the second molecule collide. This surface is that of a cylindre of revolution of the $3n-3^{\text{th}}$ degree, against the outside of which the points strike.[1]) The base of the cylindre is a sphere, the descriptive lines have here become descriptive spaces, namely plane spaces of $3n-3$ dimensions.

In the collision referred to, the extension from which the points come being large compared to the dimensions of the section of the cylindre (or at least of the same order) the infinitesimal pencil will extends in the directions of the perpendicular section of the cylindre, so here in 3 dimensions, to a pencil of finite width. If from this we take again an infinitesimal part, it comes from a definite point of each section of the sphere, and so from the points of a descriptive space of the cylindre.[2]) Part of this strikes again against another cylindre (which e.g. involves collision of the 1st and the 3rd molecule), and the infinitely narrow pencil extends again to finite width; etc.

The representing points which have not taken part in these collisions strike again against another surface, and the pencil extends every time to one an infinite number of times wider, but every time in other

[1]) If the new coordinates ξ, ξ', η, η', ζ and ζ' determined by $\xi = \frac{1}{2}(x_1 - x_2)\sqrt{2}$, $\xi' = \frac{1}{2}(x_2 + x_2)\sqrt{2}$ etc. are introduced, the equation of the surface becomes $\xi^2 + \eta^2 + \zeta^2 = 2r^2$.

[2]) Properly speaking a narrow region in the direction of this descriptive space.

spaces. However, when in a system the same molecule has had a great number of collisions, the extension has had a projection on the space determined by the axes of the coordinates of the 1st molecule for each of these collisions. So when every molecule has had a great number of collisions the above mentioned vector of velocity has passed round the sphere on which the points of velocity lie, a great many times in every direction. The points of velocity which originally covered an element of the spherical shell, will now occupy the whole spherical surface many times. As, however, the points of the sphere, where a point of velocity is after one, two etc. revolutions in a certain direction, come to the same thing with respect to the distribution of the velocities of the molecules, as in all the preceding cases we may again assume that finally the density is the same all over the spherical surface.

For the rest of the elements of the same spherical shell originally occupied the same reasoning holds. If there are also systems in the ensemble with another kinetic energy, the points disperse also here homogeneously in spherical layers; as, however, one kind cannot pass into another, the density may be different for the layers. It is the same as in the distribution of place when the gas masses are in different vessels. [1])

Now the problem of the placing of the molecules. For this purpose we consider a part of the phase-extension, originally determined with regard to place by limits lying indefinitely near each other, but occupying a finite part of the space of velocities. Now we have to demonstrate

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[1]) Also without BOREL's way of representation the above mentioned dispersion may be imagined to a certain extent. In each of a number of systems the molecules have the same velocities, but different places. Now it will entirely depend on the mutual situation e.g. of the molecules 1 and 2 in connection with their velocities, what the direction of the normal becomes in the collision, and so up to a certain extent, what the final velocities will be. In any case we get an infinitely large number from a single pair of velocities. Whereas we had before infinitely narrow limits between which the components of velocity had to lie, now we get a finite region. If we have chosen a definite one from the pairs of final velocities so also a definite velocity of the 1st molecule, this molecule may have all kinds of positions with regard to the 3rd molecule, against which it will presently strike, so also the normal of collision can have all kinds of directions, and so the limits thought infinitely narrow are again removed to a finite distance, etc. If we now take as variables the angles of the general vector of velocity with the axes of velocity instead of the components of velocity, the moving apart of the limits will yield a larger amount of occupied angles, so that finally there is occupied an amount of a large number of times 2π. If we now take into account, that increase of such an angle by an amount 2π has no effect, we arrive at considerations and results of quite the same nature as in the problems treated above.

that finally this ensemble will be uniformly scattered over all the combinations of place. In this case there will again be a partial entropy S_p which has become minimum.

We may try to make use of BOREL's way of representation also here, and shall take as introductory case an ensemble of systems consisting of 2 molecules moving along the same line perpendicular to two parallel walls between these walls.

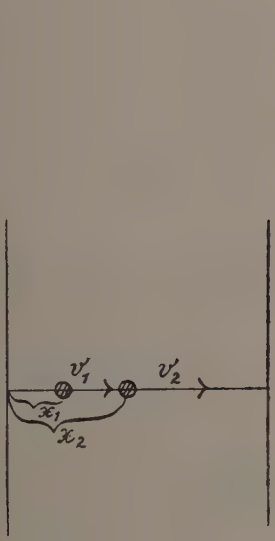


Fig. 3*a*

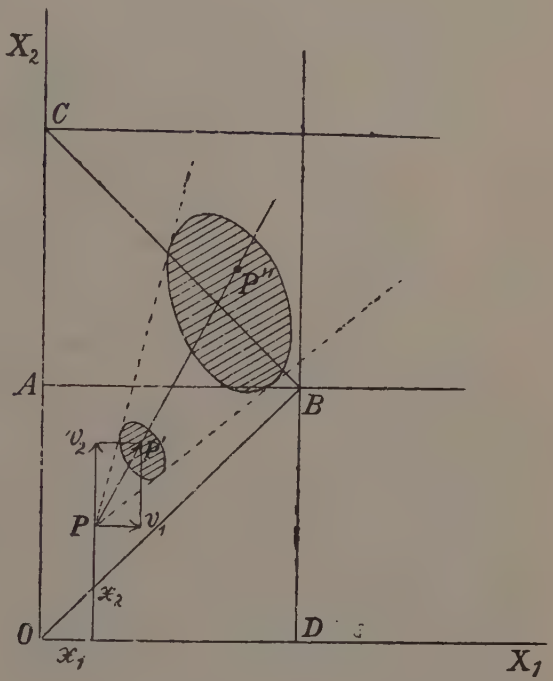


Fig. 3*b*

Fig. 3*a* represents one system from the ensemble, fig. 3*b* the motion of the whole ensemble, when the dimensions of the molecules are disregarded. Originally all the systems are in an element placed at $P(x_1, x_2)$, the points of velocity lying within an arbitrary figure. Here the representing points will be found after 1 sec., whereas they would occupy a continually extending figure if there was no collision. Collision of the molecules with the walls is here represented by collision of the representing points with the lines OA and AB, collision of the molecules inter se by collision with the line OB.

Now, however, we may also think the motion after the collision e.g. with AB continued past AB, if the triangle ABO which

would then be passed through for the second time with the velocity after impact is thought to be turned over along AB. Now the point pursues its course uninterrupted. A following collision with OB now becomes collision with BC; now we may proceed again in the same way. The pencil then goes on without any disturbance, but we must take into account that the elements of surface, which in this way proceed from each other, represent the same placing of the two molecules. The continually extending figure of the representing points will finally contain a very large number of elements of surface of every kind coming to the same thing, or a very large number of turned over triangles, so that finally the points will be uniformly distributed over the sums of the elements of surface. So every situation of the two molecules represented by a point in ABO, occurs equally frequently. A point of BDO, however, is never reached: so every situation in which the 2nd molecule is on the right of the 1st is equally probable, but the 2nd cannot get to the left side of the 1st.

Now we should have to extend this reasoning to the case of more than two dimensions. The reflection against the walls does not affect our reasoning. The striking of the molecules against each other, however, is now represented by striking against a cylindrical surface. Though this obstructs the way, it no longer shuts off a part of the space. The case may be compared with that of fig. 3$_b$ if the line OB is replaced by a circle. I have not succeeded in solving the problem for this general case. However, it seems very plausible that the finite number of cylindres will not be able to prevent that the uniform distribution of the representing points over the sums of the elements of volume which come to the same thing, which distribution would finally come about as we saw in § 3 if there was no collision, will be established also now. So all the combinations of place of the molecules would then occur equally frequently.

§ 5. Finally it may still be shown that when all the combinations of place and all the combinations of velocity occur with equal frequency, it follows from this that for the great majority of the systems the molecules are distributed about uniformly over the vessel, and have MAXWELL's distribution of velocities. So far we have always distinguished between the individual molecules, now we shall have to take into consideration, that exchange of the molecules does not affect the *distribution* of place or velocity, so far as we can know it. So all the combinations which arise from each other by exchange of molecules, now come to the same thing. Hence if of s molecules s_1 are in the first element of volume, s_2 in the 2nd etc., there are

$\frac{s!}{s_1!\, s_2! \ldots s_n!}$ combinations yielding the same distribution of place. As BOLTZMANN has shown, the denominator may be represented by $ce^{\int f(xyz) \log f(xyz)\, dx\, dy\, dz}$ by approximation, in which $f(xyz)$ represents the function of distribution of the molecules over the vessel. The integral is minimum if $f(xyz) = C$, so the number of combinations is then maximum, or the uniform distribution is the most frequently occurring one. To show that the deviation from this distribution is not large as a rule, we may examine how many combinations yield a distribution, in which instead of $\frac{s}{n}$ molecules, $\frac{s}{n} + x_1$, $\frac{s}{n} + x_2$ etc. molecules occur in the elements. This number is:

$$\frac{s!}{\left(\frac{s}{n} + x_1\right)! \left(\frac{s}{n} + x_2\right)! \ldots \left(\frac{s}{n} + x_n\right)!}.$$

By putting $s! = s^{s+\frac{1}{2}} e^{-s} \sqrt{2\pi}$ etc., we get for this, taking into account that $x_1 + x_2 + \ldots x_n = 0$:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1} \left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n}+x_1+\frac{1}{2}} \ldots \left(1 + \frac{nx_n}{s}\right)^{\frac{s}{n}+x_n+\frac{1}{2}}}.$$

Now by approximation

$$\log \left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n}+x_1+\frac{1}{2}} = \left(\frac{s}{n} + x_1 + \frac{1}{2}\right)\left(\frac{nx_1}{s} - \frac{n^2 x_1^2}{2s^2} + \ldots\right) =$$

$$x_1 + \frac{nx_1}{2s} + \frac{nx_1^2}{2s} \ldots$$

So the *log* of the denominator (with the exception of the first factor) is by approximation:

$$\Sigma x + \frac{n\Sigma x}{2s} + \frac{n}{2s}\Sigma x^2 = \frac{n}{2s}\Sigma x^2.$$

The number of combinations becomes now:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}} e^{-\frac{n}{2s}\Sigma x^2};$$

if we put $\Sigma x^2 = nu^2$, the chance, that the mean deviation is smaller than u, may be represented by:

$$c\,\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}}\int_0^u e^{-\frac{u^2}{\frac{2s}{n^2}}}du=\frac{2}{\sqrt{\pi}}\int_0^{\frac{u}{\frac{1}{n}\sqrt{2s}}} e^{-t^2}\,dt.$$

Very soon, however, this value is very large, when u is only a few times $\frac{1}{n}\sqrt{2s}$ as yet; then u or the mean x, however, is still small compared to $\frac{s}{n}$, the mean number of molecules per volume element.

In a similar way the problem of the distribution of velocities may be treated. Here the denominator of the expression $\frac{s!}{s_1!\,.\,s_2!\,\ldots\,s_N!}$ may be reduced to $Ce^{\int f(\xi\eta\zeta)\log f(\xi\eta\zeta)\,d\xi\,d\eta\,d\zeta}$, in which $f(\xi\eta\xi)$ represents the function of distribution of the points of velocity. The integral is minimum, taking into consideration that Σv^2 is constant, when $f(\xi\eta\xi)=ae^{-b(\xi^2+\eta^2+\zeta^2)}$. Now it remains to investigate what is the chance to a given deviation from this distribution. We may define this deviation by the figures $x_1, x_1'\ldots x_2, x_2'\ldots$ etc. indicating the relative surplus of points of velocity in the elements of volume, respectively with the velocities v_1, v_2 etc. In the first element is then the quantity $s_1=ae^{-bv_1^2}(1+x_1)$ [1]), in the second $s_2=ae^{-bv_1^2}(1+x_1')$ etc., so that the number of combinations to be taken together:

$$\frac{s!}{[ae^{-bv_1^2}(1+x_1)]\,![ae^{-bv_1^2}(1+x_1')]\,!\ldots[ae^{-bv_2^2}(1+x_2)]\,!\ldots}.$$

The first factor of the denominator is equal to (by approximation):

$$(ae^{-bv_1^2})^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times e^{-ae^{-bv_1^2}(1+x_1)}\times\sqrt{2\pi}=$$
$$=C_1(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1}\times(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times e^{-ae^{-bv_1^2}x_1}.$$

If we multiply by the other factors, the latter part vanishes, as $\Sigma ae^{-bv_1^2}x_1=0$. So we keep:

$$C(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1}(1+x_1)^{ae^{-bv_1^2}(1+x_1)+\frac{1}{2}}\times(ae^{-bv_1^2})^{ae^{-bv_1^2}x_1'}$$
$$(1+x_1')^{ae^{-bv_1^2}(1+x_1')+\frac{1}{2}}\times etc.$$

If we take the Nep. log. the former part vanishes, as also $\Sigma\, ae^{-bv_1^2}x_1\,.\,v_1^2=0$, so that log. denominator:

$$=\Sigma\,[ae^{-bv_1^2}(1+x_1)+\tfrac{1}{2}]\,log\,(1+x_1)+C.$$

[1]) This a is equal to the above one multipled by $d_\xi d_\eta d_\zeta$.

This is by approximation:

$$ae^{-bv_1^2}\left\{x_1 + \frac{x_1^2}{1.2} + \ldots\right\} + ae^{-bv_1^2}\left\{x_1' + \frac{x_1'^2}{1.2} + \ldots\right\} + etc. + C$$

or

$$ae^{-bv_1^2}\frac{\Sigma x_1^2}{1.2} + ae^{-bv_2^2}\frac{\Sigma x_2^2}{1.2} + \ldots + C,$$

so that the whole form may be represented by

$$Ce^{-\left[ae^{-bv_1^2}\frac{\Sigma x_1^2}{1.2} + ae^{-bv_2^2}\frac{\Sigma x_2^2}{1.2} + \ldots\right]}$$

or if we denote the most probable quantities per element by a_1, a_2 etc.:

$$Ce^{-\left[a_1\frac{\Sigma x_1^2}{1.2} + a_2\frac{\Sigma x_2^2}{1.2} + \ldots\right]}$$

This exponent agrees perfectly with that obtained for the preceding problem. We may reduce the latter to the form $-\Sigma\left(\frac{ax}{\sqrt{2a}}\right)^2$, and the former to the form $-\Sigma\left(\frac{x}{\sqrt{\frac{2s}{n}}}\right)^2$; now they represent the negative sum of the squares of the absolute deviations divided by the root from twice the normal number. Just as in the preceding problem the chance to a combination of deviations for which the root from this latter sum does not amount to more than a few times unity is now very large. If we now take as measure for the deviation the mean relative deviation or $\sqrt{\frac{\Sigma x^2}{N}}$, we see that this value is very small compared to $\sqrt{\Sigma\frac{ax^2}{2}}$, so that this mean deviation will be very slight [1]).

To conclude we may still remark that in order to get in the end both uniform distribution of the molecules over the vessel and MAXWELL's distribution of velocities, originally both a finite part of the space of velocities and of the space of coordinates must have been occupied. Or there must be such an uncertainty as to the original situation and velocities of the molecules that we must consider possible a finite whole of combinations with regard to both. This finite whole of possible combinations constitutes the ensemble, which we follow in its course instead of the system unknown within certain limits.

[1]) It having been assumed in the calculation that a considerable number of points of velocity still occur in every element, we must not think of the whole of the space of velocities when estimating the number of elements N.

Crystallography. — *"On the permissible orders of the axes of symmetry in crystallography."* By Prof. W. VOIGT at Göttingen. (Communicated by H. A. LORENTZ).

(Communicated in the meeting of November 30, 1907.)

In one of the articles of the second part of his collected papers, H. A. LORENTZ took up the question — which is equally important in both crystallography and crystalphysics — of the permissible order of an axis of symmetry of the first or the second kind. In this investigation he proceeds from the principle of the rational duplicate ratio, from which he first proves that it is consistent with itself, and therefore a suitable basis for crystallographical deductions.

The study of this interesting treatise led me to the thought, that for the purpose at hand another fundamental principle of crystallography — viz. that of the rational indices — might well form a more convenient starting point. The continuation of this thought led me to the following development, which, I believe, attains the end in view in a remarkably simple and short manner. I will prove for this useful fundamental principle, as LORENTZ did for the principle of the rational duplicate ratio, that it does not contradict itself and then derive from it the permissible orders of the axes of symmetry.

1. The principle of the rational indices, as is well known, is as follows.

If we select three arbitrary boundary surfaces of a crystal polyhedron and draw through any point O parallels to their lines of intersection to form a system of axes; if we choose further two other arbitrary positions through this system of axes, and then the intercepts of these planes upon the axes are

$$u = OA, \quad v = OB, \quad w = OC \text{ on the one hand,}$$
$$u' = OA', \quad v' = OB', \quad w' = OC' \text{ on the other,}$$

the principle of the rational indices maintains then, that,

$$\frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w} = z_1 : z_2 : z_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

forms at all times a ratio of whole numbers.

In order that this principle should lead to no contradiction, it is necessary that if one proceeds from three *other* boundary surfaces of the polyhedron and uses *their* lines of intersection as the fundamental system of axes, then the polyhedral surfaces have *also on these axes* intercepts with the above mentioned relation, if the principle held with reference to the first system of axes.

The following consideration with reference to figure 1 proves that

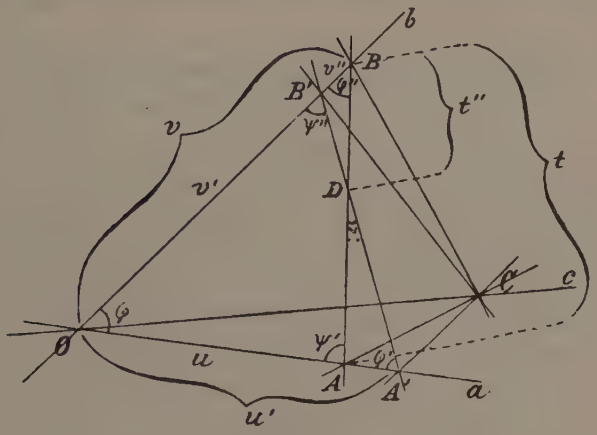


Fig. 1.

this is true. Oa, Ob, Oc form the first fundamental system of axes; ABC and $A'B'C$ represent the two planes for which the principle of the rational indices holds, i.e. their intercepts on the axes fulfil equation (1). For a simple proof it is essential that we let the two planes cut the Oc-axis in the same point, so that $w = w'$ and equation (1) assumes the form

$$\frac{u'}{u} : \frac{v'}{v} : 1 = z_1 : z_2 : z_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As second system of axes we take the lines BO, BA and BC, and as second pair of surfaces, which likewise cut the BC-axis in one point, the surfaces $A'B'C$ and AOC. If the principle is to lead to no contradiction, from (2) must follow

$$\frac{t''}{t} : \frac{v''}{v} : 1 = \zeta_1 : \zeta_2 : \zeta_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the notation to the left is explained by the figure and ζ_1, ζ_2 and ζ_3 are likewise whole numbers.

If we understand by r and ϱ rational fractions we can expect that

$$\frac{u'}{u} : \frac{v'}{v} = r \text{ may follow from } \frac{t''}{t} : \frac{v''}{v} = \varrho,$$

while at the same time

$$\frac{v'}{v} = r' \text{ follows from } \frac{v''}{v} = \varrho'.$$

The latter is apparent; for by the figure $v' + v'' = v$ and hence $r' + \varrho' = 1$, follows; consequently if r' is rational, then ϱ' is also rational.

For the former the proof follows by repeated application of the law of sines, which gives according to the figure

$$\frac{u}{\sin\varphi''}=\frac{v}{\sin\psi'}=\frac{t}{\sin\varphi},\quad \frac{u'}{\sin\psi''}=\frac{v'}{\sin\varphi'},\quad \frac{v''}{\sin\chi}=\frac{t''}{\sin\psi''},$$

while

$$\chi=\psi'-\varphi'=\psi''-\varphi'',\quad \pi=\varphi+\psi'+\varphi''=\varphi+\varphi'+\psi''.$$

From these we get

$$r=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin\varphi'\sin\varphi''},\quad \varrho=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin(\varphi+\varphi'+\varphi'')\sin\varphi}.$$

The relation between r and ϱ, is most easily obtained by determining φ' from the first formula and substituting this value in the second. We thus obtain

$$r/(r-1)=\varrho.$$

This shows that a rational r leads to a rational ϱ, which was to be proved.

The last part of the proof can still be simplified, according to a suggestion by LORENTZ, if we assume the MENELAUS' Theorem as known.

The desired proof is also given, when from

$$\frac{v'}{v}=r' \text{ and } \frac{u'}{u}=r''$$

$$\frac{v''}{v}=\varrho' \text{ and } \frac{t''}{t}=\varrho''.$$

follow.

The former of these we have considered above; relative to the latter, MENELAUS Theorem gives according to the figure

$$\frac{BD}{AD}=\frac{OA'}{AA'}\cdot\frac{BB'}{OB'},$$

i. e.

$$\frac{t''}{t-t''}=\frac{u'}{u'-u}:\frac{v-v'}{v'}.$$

Hence the rationality of u'/u and v'/v gives directly the rationality of t''/t.

2. The determination of the permissible order n of an axis of symmetry follows from any one of the fundamental principles of Crystallography, but only for the case when $n\geq 5$, because each of these principles places five similar crystallographic elements in relation. We usually so proceed, that the general property which the principle gives for the cases $n\geq 5$ is also demanded for the cases $n<5$ We can however for the latter *limited* number of cases rely upon

experience, and apply the principle only for the former *unlimited* number of cases.

Since the principle of the rational indices permits surfaces of the crystal polyhedron to be translated parallel to themselves, therefore for its application axes of symmetry of the first and second kind have exactly the same value. A difference lies only in that for axes of the second kind n must necessarily be an even number.

We start with a construction upon a sphere of unit radius, through the center of which we lay all the directions that come into consideration. (Fig. 2). Let A be trace of the n-fold axis, $P_1, P_2 \ldots P_5$ the traces of the normals of 5 related surfaces (1), (2), . . . (5) of the polyhedron, such that $\psi = 2\pi/n$. The P_h's are then designated as the *poles* of these surfaces.

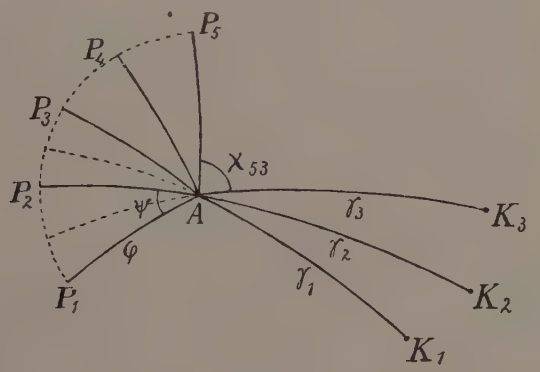


Fig. 2.

Let K_1, K_2, K_3, be the traces of the lines of intersection of the surfaces (2, 3), (3, 1), (1, 2), so that

$$K_1P_2 = K_1P_3 = \tfrac{1}{2}\pi, \quad K_2P_3 = K_2P_1 = \tfrac{1}{2}\pi, \quad K_3P_1 = K_3P_2 = \tfrac{1}{2}\pi.$$

Let K_1, K_2, K_3 form the system of axes, and (4) and (5) the pair of surfaces for the application of the principle of rational indices. It is now a question of determining the intercepts which the surfaces (4) und (5) make upon the edges K_i.

If we give the surfaces such positions that they are tangent to the sphere in their poles, then the sections σ_{hi} are identical with the reciprocals of the $\cos(K_i\ P_h)$ where $i = 1, 2, 3$, $h = 4, 5$. Consequently the values of these cosines are to be determined.

If we write γ_i, instead of AK_i, φ for AP_h and χ_{hi} for the $\angle\ P_h AK_i$ then from the $\Delta K_i AP_h$ in the figure we get

$$\cos(K\ P_h) = \cos\varphi \cos\gamma_i + \sin\varphi \sin\gamma_i \cos\chi^{hi}; \quad . \quad . \quad . \quad (4)$$

further, from the $\triangle P_3AK_1$, P_3AK_2, P_1AK_3 we get

$$tg\,\gamma_1 = tg\,\gamma_3 = \frac{ctg\,\varphi}{cos\,\frac{1}{2}\psi}\,,\quad tg\,\gamma_2 = \frac{ctg\,\varphi}{cos\,\psi}\,, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

while direct from the figure we get

$$\left.\begin{aligned} \chi_{41} &= \pi - \frac{3}{2}\psi, \quad \chi_{42} = \pi - 2\psi, \quad \chi_{43} = \pi - \frac{5}{2}\psi, \\ \chi_{51} &= \pi - \frac{5}{2}\psi, \quad \chi_{52} = \pi - 3\psi, \quad \chi_{53} = \pi - \frac{7}{2}\psi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we write the principle of rational indices

$$\frac{\delta_{41}}{\delta_{51}} : \frac{\delta_{42}}{\delta_{52}} : \frac{\delta_{43}}{\delta_{53}} = z_1 : z_2 : z_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and observe that in the quotients of each two δ_{hi}'s, since

$$1/\delta_{hi} = cos\,(K_i P_h) = cos\,\gamma_i cos\,\varphi\,(1 + tg\,\varphi\,tg\,\gamma_i\,cos\,\chi_{hi})$$

the factor standing before the brackets always cancel, then we can easily introduce the values (5) and (6) and obtain from (7)

$$\frac{cos\frac{1}{2}\psi - cos\frac{5}{2}\psi}{cos\frac{1}{2}\psi - cos\frac{3}{2}\psi} : \frac{cos\,\psi - cos\,3\psi}{cos\,\psi - cos\,2\psi} : \frac{cos\frac{1}{2}\psi - cos\frac{7}{2}\psi}{cos\frac{1}{2}\psi - cos\frac{5}{2}\psi} = z_1 : z_2 : z_3.$$

This gives directly

$$\frac{sin\frac{3}{2}\psi}{sin\frac{1}{2}\psi} : \frac{sin\,2\psi\,sin\,\psi}{sin\frac{3}{2}\psi\,sin\frac{1}{2}\psi} : \frac{sin\,2\psi}{sin\,\psi} = z_1 : z_2 : z_3 \quad . \quad . \quad . \quad (8)$$

If we now take the first and last members of this double proportion we have

$$\frac{sin\frac{3}{2}\psi}{sin\frac{1}{2}\psi} : \frac{sin\,2\psi}{sin\,\psi} = r$$

i. e. equal to a rational fraction, or also

$$\frac{1 + 2\,cos\,\psi}{2\,cos\,\psi} = r \quad \text{i. e.} \quad cos\,\psi = \frac{1}{2(r-1)} = r',$$

where r' is also rational.

This requirement, when $\psi = 2\pi/n$ and $n \geqq 5$ is fulfilled only for $n = 6$.

If we introduce this value of ψ in equation (8) we get

$$2 : \frac{3}{2} : 1 = z_1 : z_2 : z_3 ;$$

which is entirely consistent with the double proportion.

Consequently $n = 6$ *is the only value* ≥ 5 *that is consistent with the principle of the rational indices.* If we extend the requirement that *cos* ψ must be rational to the case $n < 5$ then the values $n = 2$, 3, 4, are also permissible for axes of the first kind, and the values $n = 2$, 4 for axes of the second kind.

Göttingen, November 1907.

Physics. — "*Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between* —104° C. *and* —217° C." (Continued). By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N°. 100^a from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 17. *Survey of the determinations. Remark on the apparatus.*

The measurements mentioned in this Communication comprise in the first place the supplementary determinations to which we already alluded in § 14 of Comm. N°. 99^a (Sept. 1907). These are three determinations at — 217° at a density of about 170 times the normal one.

The obvious thing to do further was to repeat the other determinations of series II with the same piezometer arranged for the determinations mentioned above, this piezometer being one of about the same dimensions as that of series II of Comm. N°. 97^a (March 1907). As a matter of fact a comparison of the values of pv_A obtained in this series with those yielded by the series III and IV teaches that the former lie somewhat, though only slightly, lower than the latter. This may be due to a systematical error as the filling in the later series was accomplished with more precautions (compare § 5 of Comm. N°. 97^a). In the series now given, just as in series IV, distilled hydrogen was used.

Both the steel tubes on the stem of the piezometer and those on the stem of the piezometer reservoir were soldered to the glass (cf. § 15 of Comm. N°. 99^a). This ensures a gas-proof connection with the steel capillary. With sealing wax it is difficult to make the connection gas-proof, because sometimes the nut begins to slide off when the flange is tightly screwed on.

§ 18. *Values of pv_A of series V.*

In table XX the results of the determinations have been represented in the same way as in table XII of Comm. N°. 97*a*. The temperatures at which the measurements were made were: —182°.74, —195°.16, —204°.62, —212°.91 and —215°.94. In table XX the reduction to the standard temperatures of table XII has been carried out. It was effected by interpolation by means of virial coefficients, which were derived in § 12 of Comm. N°. 97*a*, which enabled us to abandon the elaborate method of § 8. The computation of the temperatures took place in the same way as in Comm. N°. 95*e* (Nov. 1906). They may be reduced to the normal hydrogen scale by means of table XVIII of Comm. N°. 97*b* (March 1907).

TABLE XX, H_2. Series V. Values of pv_A.

N°.	t_s	p	pv_A	d_A
1	—182°.81	48.431	0.32746	147.90
2		55.499	0.32857	168.91
3		62.889	0.33028	190.41
4	—195°.27	42.304	0.27362	154.61
5		47.782	0.27351	174.70
6		52.808	0.27360	193.01
7	—204°.70	36.999	0.23165	159.72
8		41.258	0.23061	178.91
9		44.631	0.23001	194.04
10	—212°.82	32.035	0.19414	165.01
11		34.611	0.19270	179 61
12		37.275	0.19149	194 66
13	—217°.41	28.955	0.17318	167.63
14		31.191	0.17152	181.85
15		33.180	0.17005	195.12

§ 19. *Values of pv_A of series IV.*

If in the same way as in the preceding § the results of table XIX of Comm. N°. 99*a* are reduced to the standard temperatures the values of the adjoined table are obtained. For —139°.88 this

TABLE XXI, H_2. Series IV. Values of pv_A.

N°.	t_s	p	pv_A	d_A
1	— 103°.57	28.447	0.63261	44.967
2		38.186	0.63702	59.944
3		48.724	0.64198	75.897
4		58.368	0.64694	90.222
5	— 139°.88	25.406	0.49459	51.368
6		33.774	0.49697	67.960
7		41.273	0.49967	82.600
8		48.558	0.50232	96.667
9	— 164° 14[1])	22.818	0.40065	56.952
10		28.688	0.40164	71.427
11		34.387	0.40253	85.427
12		39.947	0.40376	98.936
13	— 182°.81	20.496	0.32704	62.670
14		24.818	0.32699	75.898
15		28.506	0.32672	87.248
16		32.568	0.32675	99.673
17	— 195°.27	18.527	0.27827	66.581
18		23.303	0.27724	84.055
19		27.837	0.27580	100.933
20	— 204°.70	16.749	0.24036	69.684
21		20.453	0.23876	85.658
22		24.015	0.23691	101.367
23	— 212°.82	15.416	0.20644	74.679
24		18.038	0.20430	88.296
25		20.643	0.20228	102.051
26	— 217°.41	14.635	0.18738	78.103
27		16.784	0.18491	90.766
28		18.853	0.18289	103.080

[1]) This temperature has been derived from the comparison of the resistance thermometer with the hydrogen thermometer of July 3, '07 (see table I, Comm. N°. 101a).

reduction has not been carried out, as it is better to take this temperature as standard temperature instead of — 135°.71. Here the reduction would have to be made for a comparatively large difference of temperature, and would become inaccurate. It is, therefore, preferable to leave the values of table XIX for this temperature intact, and if necessary apply the reduction to those of series I, which are much less reliable. The temperature — 164°.14 has been adopted as new standard temperature.

The determinations 5 and 9 as well as 14 and 18 of table XIX have been united to a mean.

§ 20. *Comparison of the series I and II with the control-determinations.*

For reasons mentioned in § 17 the points of series I and II have been doubly determined in a mutually perfectly independent way. They can be easily compared with the control determinations of IV and V by reducing them to the same density and temperature by means of virial coefficients. If in this way for —103°.57 Nos. 2, 3 and 4 of series I (see table XII) are compared with 1, 2 and 3 of series IV (see table XXI), we find for the differences of pv_A for IV—I respectively:

$$+ 0.00001 \quad , \quad + 0.00007 \quad , \quad - 0.00019$$

and for —139°.88 for IV (5,6) — I (2,3):

$$- 0.00085 \quad , \quad - 0.00036.$$

Dealing in the same way with the series II and V (see tables XII and XX), we find respectively for the temperatures —182°.81, —195°.27, —204°.70, and —212°.82,

V (1, 2) — II (2, 3) = + 0.00007, + 0.00010

V (4, 5, 6) — II (2, 3, 4) = + 0.00012, + 0.00026, + 0.00017

V (7, 8, 9) — II (2, 3, 4) = + 0. 0020, + 0.00019, + 0.00034

V (10, 11, 12) — II (2, 3, 4) = + 0.00013, + 0.00008, + 0.00021

The differences are to be ascribed chiefly to condensation of impurities, as they diminish with increase of the temperature. This was considered as sufficient ground to reject the results of the series I and II for the further calculations as less reliable. This was also done for —104°, though the series I and IV harmonize very well for this temperature. When we disregard the influence of the probable condensation the very regular course in the situation of the points is an indication about the accuracy of the measurements themselves also for the other isotherms.

So there remain the determinations of the series III, IV and V, which, reduced to the standard temperatures, occur in the tables XII, XX and XXI. With these data the further calculations have been carried out. Plate I gives a survey of the situation of the points in the diagram of isotherms; on this plate $\frac{pv_A}{T}$ has been given as function of the density. (T absolute temperature). By I and II the isotherms of 100°.20, and 0°, which will be treated in the following communication, are indicated, by the other Roman figures ascending with decrease of temperature, those to —217°.41.

§ 21. *Individual virial coefficients.*

In the same way as has been explained in § 12 of Comm. N°. 97[a] the first three virial coefficients of the development into series considered there were calculated for every isotherm, by means of the earlier and the new data. They have been put together in the subjoined table.

TABLE XXII. H_2 Individual virial coefficients.

t_s	A_A	$10^3 B_A$	$10^6 C_A$	$10^{12} D_A$	$10^{18} E_A$
—103°.57	0.62048	+0.24409	+0.5300	+0.9113	—0.648
—139°.88	0.48765	+0.11175	+0.4034	+0.6753	—0.378
—164°.14	0.39891	+0.00732	+0.4148	+0.4970	—0.208
—182°.81	0.33063	—0.07947	+0.3908	+0.3809	—0.088
—195°.27	0.28508	—0.12309	+0.3165	+0.2892	—0.016
—204°.70	0.25074	—0.17328	+0.3398	+0.2166	+0.031
—212°.82	0.22103	—0.22271	+0.3599	+0.1514	+0.066
—217°.41	0.20424	—0.24539	+0.3558	+0.1122	+0.082

In the same way as in Comm. N°. 97[a] the differences between the observed values of pv_A and those calculated by means of the found virial coefficients are put together in a table, which we subjoin.

The second column contains the differences for the points of the hydrogen thermometer (see table XII of Comm. N°. 97[a]), the following columns refer to the series IV, V and III in the order given here, the values being arranged according to the ascending densities for each series.

TABLE XXIII H_2. Deviations from the formula.

t_s	$10^5 (O_i - C_i)$									
—103°.57	— 1	+ 8	— 1	—11	+ 6					
—139°.88	— 2	+13	—15	0	+ 4					
—164°.14	0	— 3	+ 7	— 6	+ 2					
—182°.81	— 1	—15	+13	+ 3	+12	—15	— 9	+12		
—195°.27	— 4	— 2	+25	—11	—16	0	+ 8			
—204°.70	—18	+ 4	+36	+22	—22	—24	—21	+23		
—212°.82	—14	+ 3	+12	+22	— 5	—10	— 4	—22	+18	
—217°.41	—14	+13	0	+15	— 2	+ 1	— 3	—22	— 6	+18

It appears that on the whole series IV gives higher values than series V. The calculated curves may serve for the adjustment of the series mutually. Undoubtedly their points will be more reliable than those of the separate determinations. In future we shall, therefore, start from the virial coefficients of XXII.

§ 22. *Minima of* pv_A.

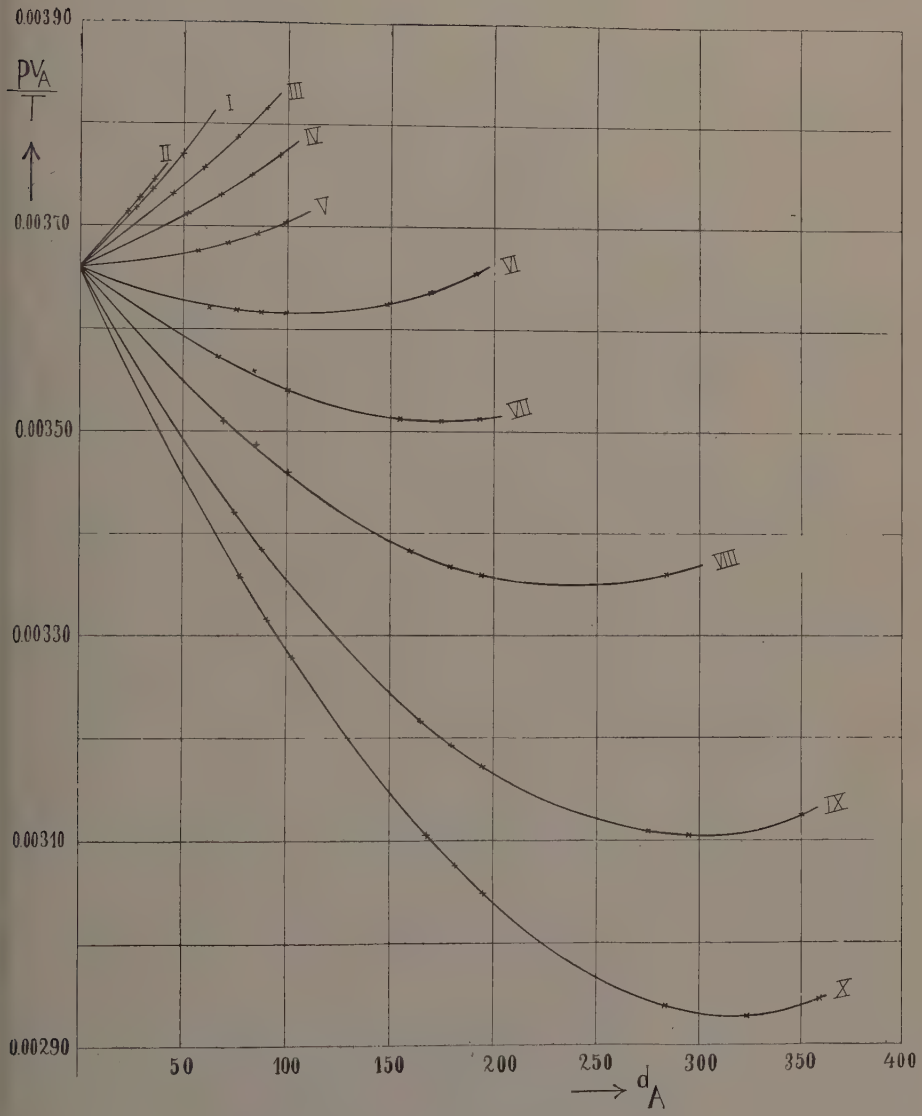
With the now available data the minima of the pv_A curves were now again determined for the lowest five temperatures, and as before the coefficients P_0, P_1 and P_2 of a parabola calculated. The columns of table XXIV have the same meaning as those of table XV of Comm. N°. 97*a*.

TABLE XXIV. H_2. Minima of pv_A.

t_s	pv_A	d_A	p	$O-C$
— 182°.81	0.32663	99.70	32.57	—0.29
— 195°.27	0.27348	183.10	50.07	+1.36
— 204°.70	0.22945	238.27	54.67	—0.21
— 212°.82	0.18782	287.99	54.09	—0.84
— 217°.41	0.16342	321.51	42.54	+0.19
	0.39292	0	0	—0.23

H. KAMERLINGH ONNES and C. BRAAK. "Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between $-104°$ C. and $-217°$ C." (Continued).

Pl. I.



For the calculation of the coefficients of the parabola a sixth point was used which has been inserted at the bottom of the table, and was obtained by means of the isotherm of — 164°.14. For this temperature the value of B_A is very slight, and by means of interpolation the BOYLE-point can be easily determined. For this is found, measured on the absolute scale:

$$\theta = - 165°.72$$

to which corresponds a value of $pv_A = 0.39292$.

For the coefficients of the parabola we find:

$$P_0 = - \quad 14.8370$$
$$P_1 = + \quad 676.563$$
$$P_2 = - 1624\ 31$$

The differences of the last column are slight, except for —195°.27. For this temperature C_A appears also to be too small (see table XXII).

Both deviations must be owing to the not quite accurate position of one or more of the points of the isotherms. From the diagram of Plate I it is already to be seen that the middle point of series IV probably lies too high.

The parabola cuts the ordinate $p = 0$ in two points where pv_A is respectively $= 0.39330$ and 0.02323, with which agree the absolute temperatures:

$$T_1 = 6.3 \qquad T_2 = 107°.5.$$

For the top of the parabola $pv_A = 0.20826$, with which corresponds a pressure of 55.61 atmospheres. From this follows for the absolute temperature of the isotherm which passes through the top:

$$T = 64°.2.$$

Physics. — "*Isotherms of diatomic gases and their binary mixtures. VII. Isotherms of hydrogen between 0° C. and 100° C.*" By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N°. 100^b from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 1. *Survey of the determinations.*

The reservoir of 5 cm.^{3} of the piezometer of series IV (Comm. N°. 99^a Sept. 1907) was replaced by one of 10 cm.^{3} With this apparatus two isotherms were determined, in ice and in vapour of boiling water. To obtain constant temperatures the same instruments were used as in Comm. N°. 60 (Sept. 1900). The water manometer (c.f. § 8 of Comm. N°. 27 (June 1896)) was read, but the difference

of pressure amounting to no more than 0.5 mM., the corresponding correction for the temperature might be neglected. For the determination of the temperature of the steel capillary 3 thermometers were suspended along the capillary. For the determination of 100° a paper screen had been applied to turn off the rising current of heated air; the spiral with cold water, however, above the wool-packing of the boiling apparatus, had been omitted. The difference of temperature between the thermometers amounting to no more than 2°, this was permissible.

The corrected indication of the aneroid barometer amounted to 765.4 mM, from which for the temperature of the boiling-point follows 100°.20.

§ 2. *Values of* pv_A.

In the subjoined table the results of the determinations have been represented. The columns have the same meaning as in table XIX of Comm. N°. 99[a].

TABLE I. H_2. Values of pv_A.

N°.	t	p	pv_A	d_A
1	0°	27.333	1.01511	26.926
2		35.602	1.02002	34.903
3		43.413	1.02505	42.352
4		50.583	1.02964	49.127
5	100°.20	30.970	1.38619	22.342
6		39.796	1.39143	28.601
7		50.254	1.39788	35.951

§ 3. *Individual virial-coefficients.*

As has been done in § 12 of Comm. No. 97[a] we may avail ourselves of the data of table I to derive the first two virial coefficients for every isotherm. On account of the small densities which occur in these measurements, in formula (1) of § 12 of Comm. No. 97[a], D_A, E_A and F_A have been put $= 0$, so that the formula reduces to:

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Only a small number of points having been given, and the densities being small, as was observed before, C_A cannot be determined with sufficient accuracy. We borrowed the values of this coefficient from Comm. No. 71 § 3, where $C_{A_0} = 0.0_6670$ and $C_{A_{100}} = 0.0_6606$.

In order to determine the course of the pv_A curves more accurately the value of pv_A was chosen for a density corresponding to that in the hydrogen thermometer of Comm. No. 60 with which (comp. Comm. No. 97b XV § 1) 0.0036629 was found for the pressure coefficient for hydrogen at 1090 mm. zero point pressure. By successive approximations this value of pv_A is to be derived by means of these determinations of isotherms. We find for 0°:

$$pv_{A\ 0°.1100\ mm.} = 1.000256$$

and for 100°.20 with the pressure coefficient 0.0036629:

$$pv_{A\ 100°.2} = 1.367373.$$

For the density $d_A = 1.44$ may be put in both cases.

Now we obtain five values of pv_A and d_A for 0°, and four for 100°.20, from which by the method of least squares the coefficients A_A and B_A of equation (1) may be determined. These values are:

For 0°:

$$A_A = 0.99924$$
$$B_A = 0.5800 \times 10^{-3}.$$

For 100°.20:

$$A_A = 1.36626$$
$$B_A = 0.8632 \times 10^{-3}.$$

For 100°.00 we may calculate from this:

$$A_A = 1.36553$$
$$B_A = 0.8626 \times 10^{-3}.$$

The differences which remain between the values of pv_A of table I and those calculated according to formula (1) with the coefficients found here are respectively:

for 0°:

$$+ 0.00018, \quad - 0.00023, \quad - 0.00028, \quad + 0.00004, \quad + 0.00029$$

for 100°.20:

$$- 0.00013, \quad + 0.00034, \quad - 0.00001, \quad - 0.00019.$$

The first value always refers to the point calculated for the hydrogen thermometer. The differences are slight, and do not or only very slightly exceed $\frac{1}{4000}$ of pv_A.

Physics. "*On the measurement of very low temperatures.* XVII. *Determinations for testing purposes with the hydrogen thermometer and the resistance thermometer.* Communication N°. 101^a from the Physical laboratory at Leiden by Prof. H. KAMERLINGH ONNES, C. BRAAK and J. CLAY.

(Communicated in the meeting of November 30, 1907).

§ 1. *Introduction.*

In communication N°. 95^e (Nov. 1906) the results of a number of measurements are recorded which show the possibility of measuring temperatures down to — 217° with the hydrogen thermometer accurately to $\frac{1}{50}$ deg. The results obtained with several fillings showed that with our measurements to — 217° this accuracy has been reached indeed. It was our plan to make also the following measurements:

1st. more testing determinations between 0° and — 217° in order to establish still better the limit of the accuracy of the temperature measurements with the hydrogen thermometer and the accuracy of the definition [1]) once for all of special temperatures by definite resistances of a resistance thermometer;

2nd. the extension of the testing determinations to measurements in liquid hydrogen;

3rd. the determination of definite standard temperatures by means of the boiling points and melting points of hydrogen, oxygen and other substances that can be easily purified. [2])

4th (comp. Comm. N°. 95^a § 1, Sept. 1906) temperature measurements with the helium thermometer, *a.* for a direct or an indirect comparison with the hydrogen thermometer, *b.* in order to get a firm basis for the determination of the lowest temperatures, especially with a view to the reduction to the absolute scale.

The investigation mentioned sub 3 and 4 has advanced a good

[1]) In investigations the reading of temperatures with a resistance thermometer will as a rule for simplicity be preferred to reading them with the hydrogen thermometer.

[2]) When we possess the fixed points meant here, the hydrogen thermometer can for calibrations be replaced by boiling point apparatus filled with pure gas and placed in the same bath as the apparatus to be calibrated. This is a great simplification in cases where only these *few special* temperatures are required. Moreover in these fixed points we have the means for a comparison between gas thermometers filled with different gases (for instance H_2 and He) or between thermometers in different laboratories.

deal. We now intend to communicate some measurements relating to 1st and 2nd.

§ 2. *Survey of the determinations.*

With regard to the controls meant sub 2nd two independent determinations have been made with entirely the same apparatus for comparison. The measurements meant sub 1st did not entirely succeed owing to a small reparation which the resistance thermometer required.[1]) These measurements, however, have thereby acquired a signification in another respect, namely as a new calibration from $-104°$ to $-259°$ of the resistance thermometer used in Comm. N°. 95*e*, they allow us to judge in how far after similar reparations, which in the long run will be inevitable, the same temperature coefficients will remain valid for the resistance thermometer.

The communicated measurements determine also with a greater accuracy a couple of temperatures (comp. however note 2 in § 3, 2°) which hitherto had not been determined with the desired reliability (comp. §§ 3 and 5)[2]).

The results have been combined in a table following below. The second and third columns contain the readings of the hydrogen thermometer and of the resistance thermometer. Those of the hydrogen thermometer are calculated in the way of N°. 95*e* (designated by t) and therefore require the corrections mentioned in Comm. N°. 97*b* (March 1907). They have not been applied here because this had not been done in any of the preceding communications and mutual comparison is thereby facilitated. The next column shows the resistances of column 3 recalculated with the factor 1.01806, which is the ratio between the resistances at 0° C. before and after the breaking of the wire. These values have been compared with formula A_I of § 6 of Comm. N°. 95*c*. The fifth column contains the deviations from this formula. The sixth column contains the differences which were to be expected according to Comm N°. 95*c*. The seventh colum contains the differences between the two resistance thermometers in Ω.

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[1]) When the resistance broke only $^1/_{60}$ of the wire was lost, yet on account of this one might allege that if the latter is not perfectly homogeneous variations in the coefficients of temperature are not entirely excluded. These are especially to be feared as a result of the new winding of the wire.

[2]) These measurements are used in table V, Comm. N°. 99*b* (Sept. 1907) at $-252°.82$ and $-255°.18$ deviating from table I, Comm. N°. 95*c* (Sept. 1906).

TABLE I. Comparison of the platinum thermometer $Pt_{I'}$ with the hydrogen thermometer.

Date 1907	Temperature according to the hydrogen thermometer	Resistance in Ω	Resistance recalculated	$O_{Pt'I'} - C_{AI}$	$O_{PtI} - C_{AI}$	$O_{Pt'} - C_{Pt}$
3 July	— 103°.671	79.131	80.551	— 0.000	— 0.023	+ 0.023
3 July	— 119°.730	70.190	71.450	+ 0.003		
3 July	— 164°.113	45.054	45.863	+ 0.047		
25 March	— 182°.352	34 492	35.111	— 0.008	— 0.029	+ 0.021
29 June	— 216°.610	14.936	15.204	+ 0.016	+ 0.028	— 0.012
19 March	— 252°.822	1.9208	1.9553	+ 2.4131	+ 2.432	— 0 019
1 July	— 252°.839	1.9243	1.9588	+ 2.4180	+ 2.432	— 0.014
19 March	— 255°.177	1.6852	1.7154	+ 2.0518		
1 July	— 258°.864	1.4522	1.4783	+ 0.4855	+ 0.199	+ 0.287

§ 3. *Results.*

In order to be able to make conclusions from table I we remark that the later determinations have been made with the same thermometer filled with distilled hydrogen [1]).

From the results of table I and some earlier determinations at the same temperatures made with the resistance thermometer before it was broken, we may derive.

1. A comparison between the indication of the resistance thermometer before and after the breaking of the wire, abstracting from the reading errors of the hydrogen and the resistance thermometer, by means of the determinations of March 25, June 29, and July 3. They give only small differences for the observations on the last two dates. Indeed if we compare the differences O—C_{AI} of table II of Comm. N°. 95

[1]) With the earlier determinations this was the case for only some among them. We may derive from former measurements (comp. Comm. 95e), that this will give no difference till — 217°, but for measurements in hydrogen this has not been proved experimentally. The method of filling by means of electrolytic hydrogen, provided it be done carefully, may be considered as perfectly satisfactory also for these temperatures. As this method, however, involves a more complicate system of auxiliary apparatus the other must be preferred with regard to the reliability.

with those of table I of this Communication, the differences for —103°, —183° and —217° [1]) respectively are:

$$+ 0.023 \quad + 0.021 \text{ and } - 0.012$$

corresponding to

$$0°.040, \ 0°.036 \text{ and } 0°.022.$$

From this we derive that down to — 217° the variations in the temperature coefficients owing to the new winding of the wire, though not imperceptible are extremely small.

2. A comparison between two resistance calibrations, for which the same hydrogen and resistance thermometers were used, in the neighbourhood of the boiling point of hydrogen by means of the determinations of March 19 and July 1. The difference is $0.0049\ \Omega = 0°.046$ and exceeds $\frac{1}{50}$ deg. which has been derived as the limit of accuracy for measurements to — 217°. This must probably be ascribed to the fact that the measurements of the resistance are less accurate because they are made with the WHEATSTONE bridge and not with the differential galvanometer [2]).

3. A comparison between the indications of the thermometer filled with distilled hydrogen and the one used before and filled with electrolytical hydrogen, by means of the determination of July 1, abstracting from the errors of observation of the hydrogen

[1]) For —217° this difference just reaches the limit of accuracy derived in Comm N°. 95*e* and for the two other temperatures the difference only little exceeds this limit.

For — 183° another reason may be given for this difference. To a certain extent it must probably be ascribed to the circumstance that the earlier determinations (of June 30 and July 6 '06) like those at — 217° of June 30 '06 must be considered as less reliable. It appeared namely during an investigation started in Dec. 1906 that the steel capillary was no longer absolutely tight and this may also have been the case when the measurements under discussion were made. The latter becomes probable when we direct our attention to the great variations of the zero during these determinations, viz 0.33 mm., to which we alluded in § 11 of Comm. N°. 95*e* without being able to explain it then.

The fault may have arisen because at the end of May '06 the thermometer has partly been taken to pieces and the capillary was bent too much. The observations made before June '06 were not influenced by this fault.

[2]) The accuracy of the WHEATSTONE bridge is perfectly sufficient for higher temperatures below 0° C. (comp. Comm. N°. 99*b* § 2 for temperatures till —217°), but owing to the disadvantageous influence of the connecting resistances, it falls short for measurements in hydrogen where the variation of the resistance becomes so small. Therefore, simultaneously with the measurements of table I in hydrogen made with the WHEATSTONE bridge, we have calibrated another thermometer *Pt*$_4$' with the differential galvanometer in order to fix the temperatures below — 217°

and the resistance thermometer and variations in the temperature coefficients of the resistance [1]).

The difference appears to be larger than we should expect after the experience made with the higher temperatures. It may be that in the measurement of May 5 '06, the first measurement made in liquid hydrogen, in the measurement of the resistance or the reading of the hydrogen thermometer a systematic error has crept in which escaped our attention. At any rate it will be necessary to repeat the calibration at these lowest temperatures.

The differences treated in this section, in so far as they go beyond the expected limit of accuracy, point partly to abnormal sources of error, partly to errors which in future may be prevented (as for instance by always measuring small resistances by means of the differential galvanometer) and it is probable that when we avail ourselves of the experience made we shall reach also for temperatures below — 217° an accuracy to 0°.02.

§ 4. In the same way as Comm. N°. 95^e § 7 the following observations, where two resistance thermometers were simultaneously immersed in the same bath, allow us to judge of the accuracy with which a temperature is fixed by a given resistance.

With Pt_I we have made an adjustment to a definite temperature at which the resistance of Pt_{III} was determined, then the temperature was changed a little and again read on Pt_I and then the resistance of Pt_{III} was determined and reduced to the first temperature.

	temp. on Pt_I	— 87°.54,	resistance Pt_{III} 103.950 Ω [2])	
new	„ „ „ reduced	— 87°.54,	„ „ 103.959	difference 0.009 Ω or 0°.014
	temp. on Pt_I	— 216°.65	17.379	
new	„ „ „ reduced	— 216°.65	17.385	difference 0.006 Ω or 0°.009

[1]) Although it is not excluded that here the variation of the temperature coefficients is larger than to — 217°, this can by no means explain the large deviation because the wire had previously been carefully annealed. Moreover it is difficult to assume that impurities of the gas in the thermometer would be the cause, for then we must accept that about 0.7 % of air has been present in the gas, which is rather impossible on account of the great carefulness observed when the thermometer is filled.

[2]) For small differences in the calibrations of Pt_{III} and Pt_V we refer to Comm. N°. 99^b where also the zero's are given. The differences result from a more accurate determination of the ratio between the arms of the WHEATSTONE bridge and of the resistance of the conducting wires. In observations which were made from 1905—1907 it appeared that the zeros had remained unchanged to less than $\frac{1}{20000}$ (comp. also Comm. N°. 99^b).

The probable error of an adjustment to the resistance thermometer appears to be equal to that of a reading on the hydrogen thermometer (comp. Comm. N°. 95^e § 7)[1]).

The following observations related to the defining of a temperature over a longer recording period.

Pt_{III} and Pt_V were calibrated immediately after each other with Pt_I, thence we derived the temperature reading according to Pt_I on Pt_{III} and Pt_V. Afterwards and adjusted to the same temperature we have compared the gold thermometer Au_0 with Pt_{III} and Pt_V immediately after each other. The readings were

temp. Pt_I on Pt_{III},	temp. Pt_I on Pt_V,	Au_0 resistance[2])
— 58°.56	— 58°.56	40.324
— 87°.43	[— 87°.50]	34.638
— 159°.07	— 159°.08	20.393
— 216°.27	— 216°.29	8.459

If we disregard the large deviation in brackets which indicate that it must be ascribed to an irregularity, it appears that the definition of a temperature by means of a single determination on a resistance thermometer has about the same probable error as a single determination with the hydrogen thermometer and is generally reliable to an amount which remains below 0°.02.

For the present the accuracy of the determination of a temperature which is kept constant during a long time by means of the hydrogen thermometer may be considered equal to the accuracy of the definition of a temperature by means of the resistance thermometer, where however the necessary readings, even when they are repeated require less time.

§ 5. According to § 3 the observations of June 30 and July 6, '06 must be rejected. They have been used for the calibration of the thermo-element and the resistance thermometer of Comms. N°s. 95^a, 95^c and 95^f (Nov. 1906) and hence the values — 217°.411 and — 182°.75 of the tables VIII of Comm. N°. 95^a, I, II of Comm. N°. 95^c and IX of Comm. N°. 94^f must be modified.

The determination of June 30, '06 at — 183° shows a perceptible difference with that of March 25, '07, the only one of the required

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[1]) In Comm. N°. 95^e § 7 we for the time being took as a starting point that the error in the adjustment of the hydrogen thermometer was negligible.

[2]) These values deviate a little from the values given in table III of Comm. N°. 95^f owing to corrections which were afterwards determined.

reliability. In observation 11 of table VIII we must therefore replace $-182°.75$ by $-182°.79$ and an analogous modification is required in the tables I, II and IX.

Instead of the values which relate to $-182°.75$ of table II we now get:

TABLE II. Values for $-183°$ instead of those of table II of Comm. N°. 97*c*.

t	Number of observations	O.-C. $_{AI}$	O.-C. $_{AII}$	O.-C. $_{B}$	O.-C. $_{C}$
-182.79	2	-0.008	$+0.048$	$+0.104$	-0.014

The agreement is thereby greatly improved, as O.-C.$_{AI}$ is changed from -0.029 to -0.008.

In the tables VIII of Comm. N°. 95*a* and IX of Comm. N°. 95*f* the three temperatures $-183°$ from which a mean has been derived, are all diminished by $0°.04$. O.-C. thereby becomes less by 0.0008 and the agreement improves in the same rate.

As to the determination of July 6, '06 the modification is very small.

In the tables VIII of Comm. N°. 95*a* and IX of Comm. N°. 95*f* no change occurs if we omit this last determination in the mean, in tables I and II it is not used.

With regard to this and to what has been said in note 1 of § 13 of Comm. N°. 95*a*, no new calculation has been made.

As a supplement to columns 2 and 3 of table I towards a complete calibration of Pt_I' we can use Pt_I in table V of Comm. N°. 99*b* with the zero 135.438 Ω (comp. note 1 of § 4), where we have to add the corrections of table XVIII of Comm. N°. 97*b*.

Physics. — "*On the measurement of very low temperatures. XVIII. The determination of the absolute zero according to the hydrogen thermometer of constant volume and the reduction of the readings on the normal hydrogen thermometer to the absolute scale.*" Communication N°. 101b from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES and C. BRAAK.

(Communicated in the meeting of November 30, 1907).

§ 1. *The determination of the absolute zero.*

D. BERTHELOT [1]) has used the observations of CHAPPUIS on the pressure coefficients between 0° C. and 100° C. and the slopes of the *pv*-lines at 0° C. and 100° C. to derive the mean relative pressure coefficient from 0° C. to 100° C. which the investigated gas would possess for densities in the state of AVOGADRO, (so we call for shortness the state in which the deviations from the law of BOYLE-GAY-LUSSAC-AVOGADRO may be neglected). In the same way we may use for this purpose the data of Comm. N°. 100b and Comm. N°. 60 (Sept. 1900).

If for the pressure coefficient of the hydrogen thermometer for a zero pressure of 1090 mm. and a density of 1.44 found in Comm. N°. 60 we derive the value 0.0036629, we may derive the pressure coefficient for the state of AVOGADRO (represented by α_{AV}) from the data of Comm. N°. 100b for B_A for 0° C. and for 100.°20 C. by means of the formula:

$$100 \times 0.0036629 = \frac{A_{A_0} \times 100\,\alpha_{AV} + (B_{A_{100}} - B_{A_0})\,\frac{1090}{760}}{A_{A_0} + B_{A_0}\frac{1090}{760}} \quad . \quad (1)$$

where we must replace the value of A_{A_0} found in § 3 of Comm. N°. 100b by a more accurate value:

$$A_{A_0} = 1 - B_{A_0} - C_{A_0} = 0.999419$$

and where $B_{A_{100}}$ has been derived from B_{A_0} and $B_{A_{100.20}}$ by interpolation [2]). Hence follows for the desired pressure coefficient

$$\alpha_{AV} = 0.0036619.$$

The reverse gives the temperature of the freezing point measured on the absolute scale. Hence:

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[1]) Sur les thermomètres à gaz. Travaux et Mémoires du Bureau international des Poids et des Mesures, T. XIII.

[2]) In formula (1) the curvature of the *pv*-lines is left out of account, which is permissible.

$$T_{0^\circ C.} = 273^\circ.08\ K$$

where K (KELVIN) stands for degrees on the absolute scale of which 100 occur between the freezing point and the boiling point of water.

The data are not sufficiently accurate to allow us to determine the last decimal to less than unity [1] [2]. The value found here agrees very well with that which may be derived from the most reliable data of other observers [3].

The method followed here of deriving the pressure coefficient for infinitely small densities by means of determinations of isotherms at pressures between 25 and 50 atmospheres is preferable to using either the data of CHAPPUIS or those of AMAGAT. It is true that in the former case the coefficient C may be neglected without error arising, but the small difference of pressure has a bad influence on the determination of B. On the contrary with higher pressures, such as with AMAGAT's determinations, the coefficients C and the higher ones have too much influence to allow an accurate derivation of the value of B. In our determinations C is of so small account that an error in the estimation of C may be neglected for the determination of B [4].

While therefore the influence of errors in C_A may be neglected we find on the other hand that the pressures are so large that an error in the pressure coefficient passes diminished to about

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[1] In discussing the isotherms we intend to refer to a small systematic difference between the isothermals of hydrogen at 20° C. according to Comm. N°. 70 (June 1901) and those at 0° C and 100 C. of Comm. N°. 100*b*. It rather points at $T0^\circ C. = 273^\circ.07\ K$.

[2] We intend to determine this value still more accurately with nitrogen and helium by means of determinations of isotherms at 0° C. and 100° C. and of pressure coefficients between 0° C. and 100° C. where we proceed according to Comm. N°. 60 (the determination of H_2 is also repeated), but as a higher degree of accuracy is wanted (designated by that now reached with the determinations of the isotherms) we now take a reservoir of 300 c.c.

[3] Comp. for this the note of § 2. XIV Comm' N°. 97*b*.

[4] If for instance in the adoption of C_A an error of 15% has been made, which with a view to the data of table XXII of Comm. N°. 100*a* probably includes the higher limit for the error for lower temperatures, this becomes only 0.0000001 which, considering that the greatest density which occurs in the determinations of Comm. N°. 100*b* amounts to about 50 times the normal one, would cause for 0° C. an error in B_A which remains below 0.000005. As such a systematic error would change the value of $C_{A.100^\circ}$ in nearly the same way, the error in this difference will be much smaller and, for instance, the error arising thence in the difference between $B_{A.0^\circ}$ and $B_{A.100^\circ}$ may be estimated at 0.000001, i. e. $^1/_5$ of the error in the absolute value of B_A. The error in the absolute zero arising from the two influences combined remains below 0°.01 C.

$\frac{1}{25}$ into the value of B_A. Hence in this way we have obtained data at a rather large difference of pressure, from which B_A may be derived unambiguously.

§ 2. *Reduction of the readings on the normal hydrogen thermometer to the absolute scale.*

By means of formula (3) of § 2 of Comm. N°. 97^b and with the new values of B'_T (comp. formula (2) of § 2 Comm. N°. 97^b) which may be derived from the data of table XXII, we have determined anew the corrections of the readings of the hydrogen thermometer of constant volume to the absolute scale. For this we have started from the individual virial coefficients and not as in Comm. N°. 97^b from a general temperature formula, because the course of the separate isotherms has now been ascertained sufficiently to render a similar previous equalization superfluous.

For B'_0 and B'_{100} we have adopted other values than in Comm. N°. 97^b. We have namely used the results obtained from direct determinations of isotherms at 0° C. and 100° C., of which the results are laid down in Comm. N°. 100^b. These values are:

$$B'_0 = 0.0005807 \qquad B'_{100} = 0.0006321.$$

for the absolute zero we have adopted

$$t = -273°.08 \text{ C.}$$

TABLE XXV. H_2. Corrections to the absolute scale.

t'_s	θ	$B'_T \cdot 10^3$	Δt_s	Δt	(O. C.)$\times 10^3$
— 103°.56	— 103°.54	+ 0.394	+ 0.017	+ 0.016	0
— 139°.87	— 139°.84	+ 0.229	0.029	0.026	— 1
— 164°.13	— 164°.09	+ 0.018	0.043	0.039	+ 1
— 182°.80	— 182°.75	— 0.241	0.056	0.051	+ 3
— 195°.26	— 195°.20	— 0.432	0.059	0.054	— 3
— 204°.69	— 204°.62	— 0.692	0.069	0.063	— 2
— 212°.81	— 212°.73	— 1.009	0.079	0.072	+ 1
— 217°.40	— 217°.32	— 1.203	+ 0.083	+ 0.076	+ 1

derived in the previous § from the same determinations of isotherms. The results have been combined in the preceding table in the manner of table XVI of Comm. N°. 97^b.

The differences with the earlier values remain, notwithstanding we have used entirely different data, far within the limits of the accuracy mentioned in § 3 (loc. cit.). The coefficients of the formula given there become:

$$a = -0.007117$$
$$b = +0.005962$$
$$c = -0.000185$$
$$d = +0.001330$$

With this formula the temperatures of the second column have been determined. The differences between the data of the last column but one and the formula are given in the last column.

For $-273°$ the new formula gives the same value as the one before of Comm. N°. 97^b i.e. $\Delta t = +0°.14$, for 0° C. and $+100°$ C. it gives $\Delta t = 0°$. For the temperatures between 0° C. and 100° C. the formula yields much larger negative values than those which BERTHELOT has derived with his equation of state (loc. cit. IX). For 20°, 50° and 80° are found: according to BERTHELOT:

$$\Delta t = -0°.00046 \qquad -0°.00053 \text{ and } -0°.00033$$

according to our formula:

$$\Delta t = -0°.0012 \qquad -0°.0020 \text{ and } -0°.0014$$

According to the general equation of state of hydrogen derived previously (comp. § 1 Comm. N°. 97^b) these values would be respectively:

$$-0°.0026 \quad . \quad -0°.0047 \quad , \quad -0°.0036.$$

Geology. — "*On the terms Schalie, Lei and Schist.*" By Mr. J. SCHMUTZER. (Communicated by Prof. C. E. A. WICHMANN).

East-Indian mining engineers, by whom almost exclusively the existing descriptions of rocks in Dutch have been written, under the influence of the German literature relating to this subject, have used the terms *schiefer* and *lei* side by side, on the whole with hardly any difference. The cause of this is the want of a fixed geological terminology in our language; a want which of late has also been felt by the "Ned. Mijnbouwkundige Vereeniging", as appears from its attempt to create such a terminology under the supervision of Prof. G. A. F. MOLENGRAAFF. The purport of this communication is to aid in solving the problem, *what* terms are best fitted to denote in Dutch those rocks which in German bear the name of *Schiefer* in the amplest sense of the word.

The starting-point is formed by the consideration, that, for the sake of clearness, different notions should be rendered by different words, and that, in the choice of terms, — as long as there is a choice — everything that might give rise to ambiguity, should be avoided.

The distinct difference which in nature exists between exclusively sedimentary slate, metamorphosed in a relatively slight measure and the finer or coarser crystalline, more or less distinctly foliated metamorphosed rocks, partly of sedimentary, however also partly of eruptive origin, is greatly discounted by the application of the term *lei* also to the latter, although the term "crystalline" be added. This explains partly, how the word *schiefer* could introduce itself into the Dutch terminology, though in a meaning that is far from being a fixed one; the want was felt of another presentive word. A bad choice, however, was made with the word *schiefer*[1]), as the latter, by taking up a place by the side of the word *lei*, must necessarily assume a more limited meaning in our language, since in the language from which it was borrowed, it occurs in different widely-diverging combinations[2]). The consequence of this is an absurd state of things, which is not to be improved by reintroducing the disused *scheversteen*[3]), which has been tried in the shorter form of *schever*[4]). In order to prevent a possible confusion of ideas, which lies concealed in the analogy with the German cognate word, this word seems to be less fitted. The corresponding English *shiver*, which has maintained itself among miners in the meaning of "flake of stone, shale, slaty débris[5])", does not occur as a scientific term[6]) and could not as such be put side by side with *shale* and *slate*.

The solution of the problem is considerably simplified by the circumstance that the English *shale* is etymologically identical with the Dutch *schalie*, a word not used in the north of Holland. Whilst the English *shale* as a secondary form of *scale* and *shell* may be directly reduced to

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[1]) Already used in most Dutch dictionaries.

[2]) As is well known also the Danish-Norwegian *skifer*, the Swedish *skiffer* is used to denote the English terms *shale* as well as *slate* and *schist*.

[3]) Plantijn, cf. E. Verwijs and J. Verdam, Middelned. Woordb., IV, 336—337; VII, 224; J. u. W. Grimm, Deutsch. Wörtb. IX, 1. sq. Kiliaen gives the word in the sense of 1°. *schalie*, 2°. *sicamb.*, i. e. the word used in Cleves for Latin *silex* (comm. of Prof. J. W. Muller).

[4]) See also Kluge, Etym. Wörtb. d. deutsch. Spr. 337; Grimm, l. c., E. Müller, Etym. Wörtb. d. engl. Spr. II, 378.

[5]) Wright, Eng. Dial. Dict. V, 392.

[6]) Cf. Archi. Geikie, Text Book of Geology, Jas. Geikie, Struct. a. Field Geology; Teall, British Petrography.

the old-English *scale, schale* [1]), this *schalie* is formed from the old-French *escaille* (Fr. *écaille*, Ital. *scaglia*), which, in its turn however, has been borrowed from Germanic [2]). Now, where these two words are closely connected both in origin and in meaning, it cannot be subject to serious objection to identify the Eng. *shale*: "applied to all argillaceous strata . . . which split up more or less perfectly in their line of bedding," [3]) with the Dutch term *schalie*.

Prof. J. W. MULLER had the kindness to communicate to me as follows [4]): As far as I can go into the question, the state of things with these two words is the following: Time out of mind *lei* in Holland, Utrecht and the (north and south) eastern provinces (cf. also Lorelei, Erpeler Lei on the Rhine, etc.), *schalie* on the other hand in the southwest: Flanders, Zeeland and South-Holland isles, have been the only word for both kinds of rocks, now more closely distinguished by geologists. Now you wish to confine the north-eastern word to the one, the south-western to the other kind. Of course this is something arbitrary, but in a scientific terminology, such unnatural and artificial distinctions are necessary, and there is nothing against it; the Flemish people will continue calling everything *schalie*, the Dutch *lei*, and there is no objection to this either". Prof. G. A. F. MOLENGRAAFF was so kind as to write to me that already at his lectures he availed himself of the term *schalie*, as an equivalent of the English *shale* [5]); a happy circumstance, which not only pleads for the weight of the grounds alleged, but will at the same time contribute much to make this term enter into general use [6]).

The English *slate* [7]) refers to argillaceous rocks, which by a metamorphosis, in which the pressure has predominated but a chemical

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[1]) MÜLLER, II. 365.

[2]) Cf. Anglo-saxon *sċeálu*, putamen, gluma; Gotic *skalja*, "Ziegel, eigentlich wohl Schindel, Schuppenartiges", KLUGE, 331; FRANCK, Etym. Woordb. d. ned. Taal, 827; MÜLLER, II, 365; KLUGE, 294, 351; VERWIJS and VERDAM, VII, 224; GRIMM, VIII, 2060—2064.

[3]) NICHOLSON cf. WRIGHT, V, 348. ARCH. GEIKIE's definition runs (op. cit. 2d edit 1885, 164), *shale*, (synon. Fr. *schiste*, G. *Schieferthon*), "clay that has assumed a thinly stratified or fissile structure," see further JAS. GEIKIE, 62. By the side of the French *argile schisteuse* we find the Ital. *scisto argilloso* (argilloscisto) and *argilla scagliosa*, lit. *kleischalie*.

[4]) Letter of 24 Dec. '07.

[5]) Dec. 16th '07: for argillaceous on other rocks "with indistinct, more or less shelly stratification."

[6]) The term *schalie* was used by the late Prof. J. L. C. SCHROEDER V. D. KOLK for the Eng. *crystalline schist*, but could not maintain itself in this meaning

[7]) Old-Eng. *slat, sclat, sklat*, old-Fr. *esclat*, Fr. *éclat*; COTGRAVE says: "*esclat*, a shiver, splinter, also a thin lath or shingle" cf. MÜLLER, II, 400—401.

change is not excluded, have got a thinly fissile structure, always parallel to the axis of the synclinal or anticlinal folds of system and consequently can make widely diverging angles with the original line of bedding) ("false bedding"). It is to be recommended, in accordance with the original [1]), now generally current, meaning of the word, to confine the term *lei* to these rocks, of which roof-slate in the best known representative. Therefore on the ground of the definition given, the above affix, apart from the well-known Rhenish devonian slates, ought to be given to the greater part of the so-called "Fleckschiefer", "Knotenthonschiefer", andalusite-, staurolite-, chiastolite slates, etc.

The English term *schist* [2]) is used for such heterogeneous rocks that the existence of this word by the side of *lei*, *slate* is fully justified. Not only does *schist* denote the more or less foliated peripherical facies of purely eruptive rocks, which often without any distinct boundary-line pass into adjoining strata of sedimentary rocks, whether these eruptive rocks occur in smaller laccolithes and dykes or in extensive intrusions or effusions (as many "trapps"); — also sedimentary rocks can by contact change into *schists*, metamorphosing agencies in strongly disturbed regions can without any distinction on a large scale change sedimentary and eruptive rocks into crystalline rocks, more or less distinctly, sometimes excellently foliated, such as gneiss, eyegabbro, amphibolite, etc. Though the term *schist* in its various shades of meaning [3]) and especially the adjective derived from it [4]), has got a much wider meaning in Romance languages, yet it finds also there, by the simultaneous use of the equivalents of *lei* and *slate*, a more limited application than in German. In connection with this it is perhaps recommendable, more particularly after English

[1]) Jas. Geikie, op. cit. 76, 220, sq.; Wright, 504; Arch. Geikie, 125—126: "In England the term *slate* or *clay-slate* is given to argillaceous, not obviously crystalline rocks possessing this cleavage structure", (syn. *argillaceous schist*, Fr. *phyllite, phyllade*; *schiste ardoise*; G. *Thonschiefer, Thonglimmerschiefer*).

[2]) Plantijn, "schalie, leye oft scheversteen", une ardoise, ardosia, scandula, cf. Verwijs and Verdam, IV, 336-337; VII, 224, by the side of which stood the meaning "rots, leisteen", (cf. also "een leye der scandalisieringhe"); old-Saxon. *leia*, rots, cf. Franck, 558; Kluge, 243. Mdl. Dutch had the word in the meaning of "Slate used for roofs", Verwijs and Verdam, IV, l. c.

[3]) From σχίζω, to split, to sever, to cleave, to divide; adj. verb. σχιστός; Arch. Geikie, 124: "a rock possessing this crystalline arrangement into separate folia is in England termed a *schist*. As Prof. Molengraaff communicated to me, this term was already used by him.

[4]) Sp. also *pizarra cristalina*.

[5]) which coincides with Germ. *schiefrig*, Dan. Norw. *skifrig*, Swed. *skiffrig*.

usage, to denote also in Dutch the *krystalline Schiefer* by means of the term *schist* (pl. *schisten*) borrowed from the Greek, and to confine one's self in the application of the derived adjective *schisteus* and the substantive *schistositeit* exclusively to this class of metamorphosed rocks.

While, in conclusion, I express my special gratitude to Prof. J. W. Muller for his kindly furnished contributions and for some valuable hints, I cannot omit tendering my best thanks to Prof. A. Wichmann and Prof G. A. F. Molengraaff for their lively interest and the support with which they obliged me.

Chemistry. — *"On the question as to the miscibility and the form-analogy in aromatic Nitro- and Nitroso-compounds"*. By Dr. F. M. Jaeger. (Communicated by Prof. A. P. N. Franchimont).

1. The following communication contains a further contribution to the kowledge of the mutual behaviour of the aromatic nitro- and nitroso-derivatives, of which several particulars were given previously. The entire miscib ility in the solid condition, and the form-analogy bordering on isomorphism, were established in the case of *p-Nitro-* and *p-Nitroso-diethylaniline* [1]), while afterwards, in a more extended paper of a more general nature [2]), the mutual comparison of *p-Nitro-* and *p-Nitrosophenol* and of *o-Nitroso-* and *o-Dinitrobenzene* was discussed. It then appeared that no *general* rule could be laid down as to the connection of the two classes of compounds.

A new pair of similar comparable substances which are interesting from more than *one* point of view, namely *o-Nitro-* and *o-Nitroso-acetanilide* were now studied. The result was different from that obtained with *p-Nitro-* and *p-Nitrosodiethylaniline* although analogy is present in *one* of the axial relations, and a solid solution of the two components appears to be possible to some slight extent.

§ 2. **Ortho-Nitro-Aceto-Anilide.**

$C_6 H_4 \underset{(2)}{(NO_2)} . \underset{(1)}{N H} . CO CH_3$; meltingpoint: 87½° C.

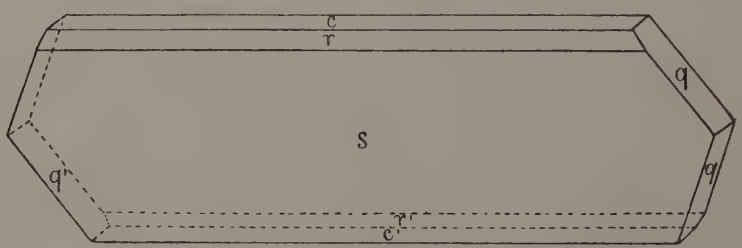

Fig. 1.

[1]) F. M. Jaeger, These Proc. VII, p. 660,

[2]) idem, Ueber Mischbarkeit von festen Phasen, Z. f. Kryst. **42**, 236—276 (1906).

On account of its great solubility in most solvents it is very difficult to obtain this compound in a properly crystallised form. The greatest success is met with by very slow evaporation of its solution in dilute alcohol when it crystallises in pale yellow, very thin laminae with extended hexagonal periphery; the crystals are very transparent and give sharp signal-images.

The symmetry is *monoclino-prismatic*. In the choice of the plane-symbols adopted here the axial relation is calculated on:

$$a : b : c = 0{,}8935 : 1 : 1{,}9198$$
$$\beta = 83°51'.$$

Forms observed: $s = \{101\}$, strongly predominant and yielding sharp reflexes; $r = \{\bar{1}01\}$ much narrower, but gives a good reflection; $c = \{001\}$, narrower than r; $q = \{011\}$, also narrow, but reflects well. The habit is flattened along $\{\bar{1}01\}$ with considerable elongation along the b-axis.

The following angular values were determined:

	Measured:	*Calculated*:
$c : q = (001) : (011) =$	* 62°21′	—
$q : s = (011) : (10\bar{1}) =$	* 80°57′	—
$c : r = (001) : (\bar{1}01) =$	* 60° 4′	—
$r : s = (\bar{1}01) : (10\bar{1}) =$	49°45′	49°45′
$s : c = (10\bar{1}) : (00\bar{1}) =$	70°11′	70°11′

A distinct plane of cleavage was not found.

In the orthodiagonal zone the direction of extinction is orientated everywhere perpendicular to the direction of the b-axis. On $\{\bar{1}01\}$ no perceptible dichroïsm is observable

The optical properties of the substance in convergent polarised light are very remarkable.

For the red, yellow and most of the green rays of the spectrum the axial plane $= \{010\}$; extraordinarily strong, inclined dispersion; the axial angle for the red is much larger than that for the green. The character of the double refraction is positive.

On the other hand, the axial angle for the blue and violet rays is situated perpendicular on $\{010\}$ with a horizontal dispersion. The axial angle for all rays is but small.

The curious colourphenomena in white light exhibited by this substance, which thus possesses at the same time inclined and horizontal dispersion, lend themselves particularly well to the demonstration of anomalous dispersion in biaxial crystals.

The sp.gr. of the crystals is 1.419 at 15°; the equivalent volume is therefore 126.85.

Topic parameters: $\chi : \psi : \omega = 3{,}7578 : 4{,}2058 : 8{,}0744$.

§ 3. Orthonitroso-Aceto-Anilide.

$C_6H_4(NO)_{(2)} . N_{(1)} H\text{-}COCH_3$; m.p.: 108° C.

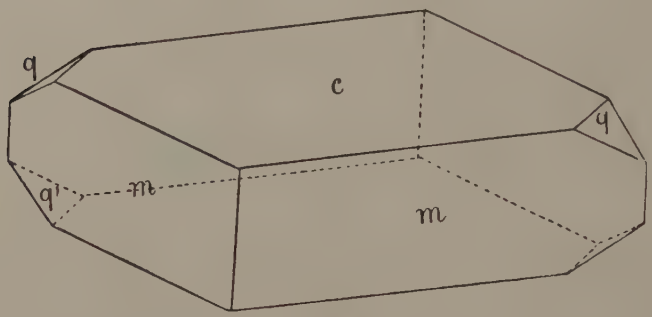

Fig. 2.

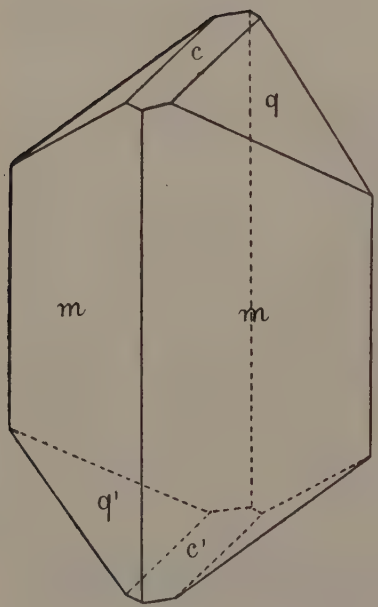

Fig. 3.

Through the kindness of Dr. F. Leuchs of *München*, who was the first to prepare this substance (Ber. 1907 40. 1083), I received a small quantity of the crystallised compound obtained by cooling the hot alcoholic solution. The crystals exhibited the habit of fig. 2; from a mixture of ether and benzene I obtained the thick prismatic crystals of fig. 3. The prism-planes, which often act as resting planes of the crystals in the mother liquor, were in consequence mostly curved and unsuitable for accurate measurement, whilst the forms c and q always gave ideal reflexes. The crystals have a brilliant emerald green colour and are quite transparent.

The symmetry is *monoclino-prismatic*; the axial relation was calculated on:

$$a : b : c = 0{,}8940 : 1 : 0{,}7295$$

$$\beta = 82° 6'$$

Forms observed: $c = \{001\}$, predominant in the crystals obtained from alcohol and always very shining; $m = \{110\}$, well developed but often with curved plane; $q = \{011\}$, giving ideal reflexes and mostly exhibiting rather large planes; $a = \{100\}$, exceedingly narrow and dim. The habit is flattened along c or long prismatic parallel

the c-axis, with flattening along two parallel planes of m; perfectly cleavable towards {001}.

The following angles were measured:

	Measured:	*Calculated:*
$c : m = (001) : (110) =$	* 84° 5¼′	—
$c : q = (001) : (011) =$	* 35° 51′	—
$m : m = (110) : (1\bar{1}0) =$	* 83° 3′	—
$m : q = (110) : (011) =$	61° 55½′	61° 51½′
$m : q = (110) : (0\bar{1}1) =$	72° 18′	72° 15′
$m : a = (110) : (100) =$	41° 34′	41° 31½′

On {001} not observable but on {110} very distinctly dichroïc; for vibrations parallel to the c-axis, grass-green; and yellowish-green for vibrations perpendicular thereon. The angle of extinction is very difficult to determine; it amounts to about 12° in regard to the c-axis on the planes of {110}.

The optical axial plane is {010}; on {001} one axis is visible at a small angle with the normal to that plane; the inclined dispersion is extraordinarily strong: $\varrho < v$.

The sp. gr. of the crystals is 1.351 at 15°, the equivalent volume is 121.39.

Topic parameters: $\chi : \psi : \omega = 5.1206 : 5.7277 : 4.1784$.

4. A small addition of the nitroso-compound to the nitro-compound causes a perceptible depression of the melting point of the latter substance. As, therefore, no certainty is obtained as to the formation of mixed crystals of these closely related derivatives — for a melting curve with an absolute minimum might be also present — a few preliminary quantitative experiments were carried out, which showed that we have here indeed an ordinary binary meltingpointline with eutecticum. It was further shown by more detailed microscopic tests that from mixed fusions of the two compounds is always deposited a *mixture* of the *yellow* nitroaceto-anilide crystals and the *green* nitrosoaceto-anilide crystals with a totally different aspect. At the side of the nitroso-compound a perceptible quantity of the nitro-derivative is carried down by the deposited crystals as solid solution; at the side of the nitro-derivative a formation of solid solutions could not be ascertained in this way. In any case, if there should be a slight mixing, it is limited on the side of the nitroso-compound to a few percent of the nitro-compound; the hiatus is therefore enormously extended.

Finally be it observed here that both substances are very volatile; the vapour of the nitroso-compound is *yellowish green*.

A few experiments were tried to sublime mixtures of the two compounds. The deposit on the concave coveringglass then consists of a network of dentritic, strongly pleochroïc (colourless — yellowish-green) little crystals, between which are found the square-shaped crystals of the nitroso-compound, besides the yellow individuals of the nitro-compound, united in clusters.

The first mentioned crystals contain chiefly the nitroso-compound and probably to a very small amount also the nitro-compound, so this may be a new case of the formation of solid solutions by sublimation. Probably, this pair of compounds lends itself to the measuring of the vapour tensions of these solid solutions.

Zaandam, December 1907.

Physics. — "*Observation of the magnetic resolution of spectral lines by means of the method of* FABRY *and* PEROT." By Prof. P. ZEEMAN.

The interference method of the parallel semi-silvered plates, worked out with so much ingenuity by FABRY and PEROT[1]) excels above all other spectroscopic modes of procedure by the accuracy with which its theoretical foundations may be practically realized.

The principal task of the experimenter in applying this method has become to effect the perfect parallelism of the reflecting silvered plates.

In order to test by an independent method some recent results obtained in an investigation of the magnetic resolution of spectral lines[2]) the method of FABRY and PEROT seemed most appropriate. Especially it appeared possible to extend at the same time the investigation to the behaviour of the lines in weak fields. The present paper is preliminary to a discussion of numerical results. I think it beforehand very improbable that errors of ruling of the ROWLAND grating will turn out to be the reason of the asymmetrical resolution of some lines, which I have described.

The method of FABRY and PEROT is applied in the present paper for the first time to an investigation of the magnetic separation of spectral lines. In some places in the literature of the subject the opinion is expressed that the method of interference fringes of silvered layers cannot be used for the subject under review. The

[1]) FABRY et PEROT, Ann. de Chimie et de Physique 1899—1904.

[2]) ZEEMAN, These Proceedings, November 1907.

main objection is derived from the great loss of intensity in the apparatus of FABRY and PEROT. The present paper proves that this objection is not insuperable.

2. Of the two forms in which the method of the parallel plates is employed, the simplest, also requiring the least costly apparatus, has been used for the actual measurement of wave-lengths by FABRY and PEROT[1]), lord RAYLEIGH[2]) and EVERSHEIM[3]). This form of instrument called *étalon*, I also have preferred. The distance of the silvered surfaces is here constant. The glass plates are held up to rounded distance-pieces (made of steel), by adjustable springs, which permit to regulate the pressure. By variation of the pressure the steel and the glass can be deformed in an extremely small degree and the accurate parallelism of the glass plates effected; the parallelism being secured already very approximately beforehand by the accuracy or finish of the distance-pieces.

3. The theory of the comparison of wave-lengths by means of this apparatus is simple, and has been given by FABRY and PEROT. We will apply it to the magnetic resolution of spectral lines and especially to the most simple case, the division into a triplet.

Let λ_0 be the wave-length of the original line (afterwards therefore the middle line of the triplet). To this corresponds a system of rings; let P_0 be the ordinal number of the first from the centre. The ordinal number at the centre p_0 is then the integer number P_0, augmented with a fraction ε_0, hence $p_0 = P_0 + \varepsilon_0$. Ordinarily $0 < \varepsilon_0 < 1$.

The diameter of a ring increases with ε. Let e be the thickness of the plate of air, the order of interference at the centre is $p_0 = \frac{2e}{\lambda_0}$. At an angle i with the normal to the plate the order of interference is $p_0 \cos i$.

If x_0 be the angular diameter of the ring P_0, then we have, observing in the focal plane of a lens, $p_0 \cos \frac{x}{2} = P_0$. Developing the cosinus

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[1]) FABRY et PEROT, Ann. de Chim. et de Phys. T. **25**, Janvier 1902. C.R. 27 Mars 1904. FABRY et BUISSON, C. R. 16 Juillet 1906. BARNES Astrophysical Journal. Vol. 19. p. 190. 1904.

[2]) Lord RAYLEIGH, Phil. Mag. Vol. 11, p. 685, 1906.

[3]) EVERSHEIM, Zeitschr. f. wissenschaftl. Photographie, Band **5** p. 152, 1907.

$$p_0 = P_0\left(1 + \frac{x_0^2}{8}\right)$$

or

$$\varepsilon_0 = P_0 \frac{x_0^2}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let λ_r be the wavelength of the outer component of the triplet towards the red then, if P_r, ε_r and x_r have a significance corresponding to that of P_0, ε_0 and x_0,

$$\varepsilon_r = P_r \frac{x_r^2}{8}.$$

We have however $\lambda_0 (P_0 + \varepsilon_0) = \lambda_r (P_r + \varepsilon_r)$, whence

$$\lambda_r = \lambda_0 \frac{P_0}{P_r}\left(1 + \frac{x_0^2}{8} - \frac{x_r^2}{8}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In like manner, λ_v, P_v, x_v determining the component of the triplet towards the violet, we have

$$\lambda_v = \lambda_0 \frac{P_0}{P_v}\left(1 + \frac{x_0^2}{8} - \frac{x_v^2}{8}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the case of radiation in a magnetic field this expression may often still be simplified. In many cases we may choose

$$P_0 = P_v = P_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Looking at the system of rings corresponding to λ_0, while the magnetic force is slowly but gradually increased one sees at the same time rings which proceed from the system λ_0 and are moving outwards and others which are moving inwards. The rings corresponding to λ_r are contracting, those corresponding to λ_v are expanding.

It depends upon the value of p of the étalon and upon the intensity of the magnetic field how great the expansion and contraction of the rings, in comparison with the distance of the rings λ_0, will be.

The value of p and the maximum magnetic force will determine whether in the centre new rings will appear or respectively will disappear.

In the case one does not select for measurement the smallest rings but if the rings λ_r and λ_v, *which originate from the same ring* λ_0, are suitable to be measured, ε can become larger than unity.

When we select the rings thu sspecified the equality (4) applies and then we may determine λ_r and λ_v from the angular diameters of the rings and the value of λ_0, regarded as known; the result is then

independent of the accurate value of the thickness of the plate of air.

Of course the position of the new rings between the rings λ_0 will, with a given value of the magnetic force, be determined by the thickness of the plate of air and what might be called "the sensibility" of the system of rings to magnetic forces will increase with the thickness of the plate of air. A limit of this sensibility is (often too soon) attained by the effective width of the spectral lines under consideration.

In some cases it will be desirable to select for measurement rings different from the three specified ones. There are no difficulties about the significance of P; it always means the ordinal number of the measured ring.

However if P_0 differs from P_r or P_v, their values must be known for the calculation according to (2) and (3).

4. Besides the simplification resulting from equation (4) there is still another one to be considered in the investigation of the radiation in a magnetic field.

I mean that the quantity $e = \mu \frac{\lambda}{2}$, the optical thickness of the plate of air may be treated as an absolute constant.

Ordinarily this thickness depends upon λ. In consequence of the change of phase by reflection upon the silver, which varies with wavelength, the comparison of different coloured systems of rings finally requires a knowledge of the optical thickness for each separate colour.

It is clear that in the application to the subject now under review only systems of rings corresponding to rays differing extremely little in wavelength are considered, hence the variation of thickness with wavelength needs not to be taken into account.

5. Figures 1 and 2 may give an idea of the aspect of the magnetic resolution of the spectral lines observed by means of the method of FABRY and PEROT. They are about sixfold enlargements of negatives taken with an étalon with an interval of nearly 5 m.m. between the plates. The source of light in the magnetic field was a small vacuum tube charged with mercury. The order of interference at the centre for the mercury line 5791 is at 16° about 17265,7.

The system of rings was formed in the focal plane of a small achromatic lens of 18 m.m. aperture and of 12 c.m. focus. Its focal plane coincides exactly with the plane of the slit of a small

spectroscope. When the slit is opened widely each spectral line is seen as a rectangle with bright rings or parts of rings as the case may be. The part of the spectrum in the figures refers to the two yellow and the green mercury lines. In fig. 1 the two rectangles corresponding to the two yellow mercury lines are superposed. The green mercury line is largely overexposed. I have reproduced it also in order to give an idea of the dispersion used. The intensity of the magnetic field in figures 1 and 2 was about 5000 Gauss.

It is a very beautiful sight to watch the moving system of rings, while the magnetic force is slowly increased. The rings λ_r and λ_v are first seen approaching, then coïnciding, separating, coïnciding for a value of the field of about 15000 Gauss with the next ring λ_0, passing over this ring, etc.

For measurements it is necessary to reduce the width of the slit, as in Fig. 2. Owing to rise of temperature the rings have somewhat expanded.

6. For measurements, which I hope to communicate in a future paper, I have used not only the method of diameters resumed above (§ 3) but also *the method of the coincidences* [1]) for the distinct values of the magnetic force, which bring to coincidence λ_r and λ_v, or λ_r and λ_v with λ_0.

Concerning the difficulties attending the use of the method of coïncidences FABRY and PEROT [1]) remark:

"Même avec ce perfectionnement, la méthode présentait des inconvénients assez graves :.

1. La necessité d'éclairer simultanément l'appareil par les deux sources entraîne des pertes de lumière assez importantes ;

2. Les coïncidences ne sont bien observables que lorsque les deux systèmes d'anneaux ont des éclats comparables, et cette condition n'est pas toujours facile à réaliser ;

3. La recherche de la coïncidence entraîne toujours des tâtonnements et l'on n'est jamais sur (lorsque la période est courte) d'en rencontrer une qui soit exacte."

The drawbacks to the method, mentioned sub 1. and 2. are eliminated in the application to radiation in a magnetic field, now under consideration. By variation of the current in the electromagnet the coincidence can be attained with the desired degree of accuracy and hence also the third objection is obviated.

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[1]) FABRY et PEROT, Ann. de Chim. et de Phys. p. 12, T. **25**, Janvier 1902

1. The yellow mercury lines in magnetic field. Very wide slit. Green mercury line overexposed.

2. The same lines. Narrow slit for measurement of the yellow mercury lines.

7. Some further details concerning the apparatus may finally be given. The mounting and the plates of the 5 m.m. étalon are by JOBIN. The inner surfaces of the plates are accurately flat. The outer surfaces need only ordinary flatness, they are inclined at an angle of 1′ to the inner ones. The plates of the étalon are vertical, and the whole apparatus is capable of the necessary adjustments in azimut, while also a horizontal sliding motion parallel to the plates of the étalon was provided for.

An image of the vacuum tube was focussed upon the étalon by means of an achromatic lens of 12 cm. focus, the enlargement being four times. All optical pieces were mounted upon double *T*-pieces and therefore rigidly connected.

The figures clearly indicate that for the investigation of the magnetic separation of the yellow mercury lines, it would be of no value to use an étalon of greater optical thickness of the plate of air. On the contrary the effective width of the yellow mercury lines when under magnetic influence is rather large, so that the limits of the method in this case are being rapidly approached.

Physics. — "*Isotherms of monatomic gases and their binary mixtures. I. Isotherms of helium between* $+100°$ *C. and* $-217°$ *C.*" Communication N°. 102ª from the Physical Laboratory at Leiden. By Prof. H. KAMERLINGH ONNES.

§ 1. On account of the important rôle, which VAN DER WAALS' theory plays in many chapters of thermodynamics, experimental data concerning the equation of state of a substance are of the greater value as the interaction of the molecules of this substance conforms the better to the hypotheses from which VAN DER WAALS started. The knowledge of the equation of state of the monatomic gases, whose molecules we must consider as the simplest for the present, is of the greatest importance from this point of view.

In Comm. N°. 69 (April 1901) on the isotherms of diatomic gases and their binary mixtures it was already observed that the investigation of the net of isotherms of argon and of helium promised still more important results than the completion of the net of isotherms of the gases formerly called permanent, particularly of hydrogen, at low temperatures, on which subject my attention had been chiefly fixed since the establishment of the cryogen laboratory (cf. Comm. N°. 14, Dec. '94). But the difficulty of obtaining argon and helium in so pure a state and in such quantities as are required for

the determination of isotherms, retarded the determination of the equation of state of helium and argon for a long time after Comm. N°. 69.

The investigations on the isotherms of hydrogen are in progress, and yielded already results laid down in communications N°. 78, 97^a, 99^a and 100^a, which, I hope, will soon be followed by others. In the meantime, however, also the difficulty of obtaining pure helium has been quite, that of obtaining pure argon nearly overcome. The successful preparation of pure helium was chiefly due to the hydrogen circulation (Comm. N°. 94f) yielding the required liquid hydrogen. So the first measurements from the series which will refer to the monatomic gases and their binary mixtures, can already be communicated.

They concern the isotherms of helium, which have now taken the place occupied by the isotherms of hydrogen before the hydrogen was liquefied. Among others the isotherms must lead to the calculation of the critical quantities for helium. From the now communicated determinations of the compressibility along different isotherms at densities which are comparatively small and differ only slightly, the critical temperature can already be calculated by approximation.

§ 2. *Survey of the determinations.*

This investigation comprises some six determinations of isotherms. The temperatures at which they were made, were kept constant and determined in the same way as in the determinations of isotherms for hydrogen published in preceding Communications N°. 97^a (April 1907), N°. 99^a (Sept. 1907), N°. 100^a (Jan. 1908). The readings of the hydrogen thermometer were reduced to the absolute scale by means of formula (4) of Comm. N°. 97^b (Sept. 1907) with the new coefficients of § 2 of Comm. N°. 101^b. The six temperatures thus reduced to which the isotherms refer, are:

$+100°.35$, $20°.00$, $0°$, $-103°.57$, $-182°.75$ and $-216°.56$.

Besides the measurements at the two standard temperatures 0° C. and 100° C.[1]) and those at low temperatures a determination was made at 20° C. to obtain data for the calculation of the quantity of gas in the stem of the piezometer and in the other parts, which remain at the ordinary temperature during the measurements.

For all these isotherms the densities, at which the pressure was observed, lie about between the same limits which were set by the

[1]) The results at 0° C. and 100° C. are incompatible with those of RAMSAY and TRAVERS, which, indeed, show strange deviations.

dimensions of the piezometer and by the manometer. The utmost limits of the density are 25 and 54 times the normal one. The piezometer and further auxiliary apparatus were perfectly the same as have served for the determinations of C. BRAAK and me (see Comm. N°. 100^b Jan. 1908) with hydrogen at 0° C. and 100° C. The satisfactory results obtained then, enhance at the same time the reliability of the measurements considered now.

§ 3. *Results for* pv_A.

The subjoined table contains the results of the determinations. The first column gives the number of observation, the second the temperature measured above 0° C. on the absolute scale, the third the pressure in atmospheres, the two following ones the product pv_A, and the density d_A, in which the volume of the gas v_A is expressed in the normal volume (that at 0° C. and 1 atmosphere) and the density d_A in the normal density (that at 0° C. and 1 atmosphere). (Compare the corresponding tables of the above mentioned determinations of isotherms of hydrogen).

The calculation of these results was made as follows:

First the points of the isotherm of 20° C. were calculated, (cf. also § 8 of Comm. N°. 79 (April 1902)) and the coefficients A_A and B_A of the curve

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

were determined by the 3 points by the aid of the method of least squares. For C_A a definite value was assumed, the densities being too small for this coefficient to be determined with sufficient certainty. If we write VAN DER WAALS' equation with the second correction for the size of the molecules in the form:

$$pv = RT + \frac{RTb - a}{v} + \frac{5}{8}\frac{RTb^2}{v^2},$$

where v is the volume of the gas under the pressure p at the absolute temperature T, expressed in the theoretical normal volume (see Comm. N°. 71 § 3) and if we put the value of A_A at 0°, $A_{A_0} = 1$, which approximation is allowed for our purpose, we find (cf. Comm. N°. 71 § 3) for the value C_{A_T} of C_A at T

$$C_{A_T} = {}^5/_8\, RT\, b^2,$$

where $R = 0.0036619$. The value of b was first estimated at 0.0005 (cf. the note to § 6 of Comm. N°. 96^c Jan. 1907), afterwards at 0,000432, see § 4. With the coefficients A_A and B_A obtained for 20°

TABLE I. He. Values of pv_A.

N°.	ϑ	p	pv_A	d_A
1	+ 100°.35	42.574	1.38725	30.689
2		54.459	1.39314	39.091
3		66.590	1.39929	47.589
4	+ 20°.00	27.539	1.08664	25.343
5		36.303	1.09028	33.297
6		53.708	1.09918	48.862
7	0°	26.634	1.01392	26.268
8		38.565	1.01851	37.864
9		50.240	1.02521	49.004
10	— 103° 57	20.580	0.63135	32.597
11		24.100	0.63296	38.075
12		29.185	0.63597	45.891
13		33.383	0.63845	52.288
14	— 182° 75	13.751	0.33787	40.699
15		16.019	0.33898	47.257
16		18.189	0.34025	53.457
17	— 216°.56	9.564	0.21132	45.259
18		10.502	0.21171	49.606
19		11.448	0.21219	53.951

in this way, the reductions to 0° for the gas which is outside the reservoir at a temperature of 20° and for a small part at the temperature of the room, were carried out in first approximation. With the three points which were thus found on the isotherm of 0°, the virial coefficients A_A and B_A were then calculated also for this temperature.

From this follows A_{A_0}, the value of pv_A for $d=0$ by means of the formula:

$$A_{A_0} = 1 - B_{A_0} - C_{A_0}.$$

With the pressure coefficient from 0° C. for the state of AVOGADRO, 0.0036619 (cf. § 1 of Comm. N°. 101^b) follows for 20°:

$$A_{A_{20°}} = A_{A_0}(1 + 0.0036619 \times 20),$$

so that on the isotherm of 20° a fourth point is acquired, which renders the slope of the pv_A line a great deal more certain (cf. the conclusion of § 1 of Comm. N°. 101^b). Then the calculation of A_A and B_A was repeated, and by the aid of these corrected coefficients the isotherm of 0° C. was again calculated, and this calculation by approximation was continued till it caused no longer any change. In this way we found for 20° C. (for b=0.000432):

$$pv_{A_{20}} = 1.07273 + 0.0005337\, d_{20} + 0.000000125\, d_{20}^2 \quad . \quad . \quad (3)$$

With this formula the corrections have been calculated for the determinations of isotherms. For the rest the latter were treated as in the preceding communications.

§ 4. *Individual virial coefficients.*

We may avail ourselves of the data of table I in order to derive the coefficients A_A and B_A by the aid of the method of least squares. C_A was assumed according to formula (2) of the preceding §. For every isotherm $pv_{A,d=0}$ was calculated, and this value was added to the others as if it concerned a new observed point. This calculation was effected by the aid of the value $A_{A_0} = 0.99949$, which may be derived from the value for the coefficients B_A and C_A for 0° finally obtained in the calculation by approximation from the conclusion of the preceding §. Table II contains the virial coefficients and at the same time the differences between the given pv_A's and the calculated ones. These differences are arranged according to the ascending densities. So the first column of differences refers to $pv_{A,d=0}$, the others to the data of table I in the above succession.

The calculation of the B_A's is still uncertain, because for C_A estimated values have been assumed. Determinations of pv_A at greater densities, which will render an independent determination of C_A possible, are in preparation.

That the estimations of C_A are not too inaccurate, may be made probable as follows. For 100° follows from table II $B_{A\,100°} = 0.000673$. On the suppositions on which VAN DER WAALS' equation rests, the value of b may be derived from the value for two temperatures of $B = RTb - a$ with $B = B_A$ (A_{A_0}) and then $b = 0.000432$ is found, which does not differ much from the value 0.0005, which was first assumed by way of estimation on other grounds. Though the calculation followed here is very uncertain, yet the found value was preferred to the first estimated one, and for this reason the calculations which were first made with 0,0005, have been repeated with this new estimation. The differences of the results lie within the limit of errors of observation.

TABLE II. He. Individual virial coefficients.
Deviations of the pv_A from the calculated ones.

t	A_A	$10^3 B_A$	$10^6 C_A$	10^5 (O—C)				
+100° 35	1.26667	+0.673	+0.16	+10	−21	−10	+22	
+ 20°.00	1.07273	+0.534	+0.13	− 3	+31	− 36	+ 8	
0°	0.99970	+0.512	+0.12	−20	+80	−75	+15	
−103°.57	0.62026	+0 337	+0 07	+11	− 7	− 32	0	+27
−182°.75	0.33066	+0 176	+0 04	1	− 1	− 8	+ 8	
−216°.56	0.20693	+0.096	+0 02	0	+ 1	− 3	+ 3	

§ 5. *Determination of the critical temperature of helium.*

From the data of table II we may arrive at a first estimation concerning the critical temperature of helium, which will be found from determinations of isotherms within the now accessible region of temperatures.

Extrapolation proves that the BOYLE-point will lie in the neighbourhood of — 250° C. For hydrogen — 166° was found for this. (cf. Comm. N°. 100^a). If we assume 30° K. for the critical temperature of hydrogen, then follows from this for helium

$$T_{kHe} = 6° K.$$

If this value of T_k is adopted, the region of temperature — 217° to — 183° for helium corresponds with that of 0° to + 200° for hydrogen. By applying the law of the corresponding states to the slopes of the pv_A-lines for the two substances in the neighbourhood of these equivalent limits of temperature, we arrive at a slightly lower value of the critical temperature viz.

$$T_{kHe} = 5°.3 K.$$

This value too I think I may still consider as a highest limit for the critical temperature of He, as it seems probable to me, that He with respect to H_2 will deviate from the law of the corresponding states in this sense that the critical temperature will be found lower than would follow from the application of this law to corresponding states for values of the reduced temperatures many times larger than 1.

Now there can only be question of a first estimation based on determinations of isotherms. The determination of the isotherms of — 253° and — 259°, which is in progress, will, I hope, soon lead to a more reliable estimation.

In conclusion I gladly express my thanks to Mr. C. BRAAK for his assistance in this investigation.

(January 23, 1908).